Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Department of Physics

Emergent gravity

Emergentní gravitace

Research Project

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Academic year: 2023/2024

FAKULTA JADERNÁ A FYZIKÁLNĚ INŽENÝRSKÁ
KATEDRA FYZIKY

VÝZKUMNÝ ÚKOL

Akademický rok: 2023/2024

Student:	**Bc. Jan Masák**
Studijní program:	Matematická fyzika
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Název úkolu (česky):	Emergentní gravitace
(anglicky):	Emergent gravity
Jazyk VÚ:	angličtina

Pokyny pro vypracování:

1) Seznamte se s holografickým principem v klasické a kvantové gravitaci.
2) Seznamte se s termodynamickou formulací gravitace. Speciální důraz bude kladen na Jacobsonovo odvození Einsteinovy gravitace z termodynamických principů a na Verlindeho entropickou gravitaci.
3) Diskutujte důsledky zobecněné termodynamiky (Tsallisovy, Borowovy či Rényiho) na emergentni gravitaci.
4) Diskutujte získané výsledky.

Literatura:

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Datum zadání: 20.10.2023

Datum odevzdání: 15.08.2024

Výzkumný úkol odevzdá student v elektronické podobě nahráním souboru ve formátu pdf do příslušné události v systému Indico. Vedoucí úkolu přiloží své písemné vyjádření k práci studenta s doporučením pro hodnocení.

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Acknowledgment:
I would like to thank prof. Ing. Mgr. Petr Jizba, Ph.D. for his expert guidance, patience, time, and invaluable insights. Furthermore, I would like to thank my family for supporting me to the best of their abilities throughout my studies and my friends for creating an excellent environment at the university.

Název práce:

Emergentní gravitace

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Studijní program: Matematická fyzika

Druh práce: Výzkumný úkol

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Abstrakt: Tato práce se zabývá myšlenkou, že gravitace může být emergentním jevem, nikoli fundamentální interakcí. Diskutujeme motivace, které stojí za tímto pohledem, a také rámce, ve kterých byl tento pohled rozvinut. Náš přehled začíná termodynamikou černých děr, včetně čtyř zákonů, Hawkingova záření a zobecněného druhého zákona. Následně se obracíme k holografickému principu, se zaměřením na entropické meze — zejména na Boussoovu mez — a diskutujeme jeho realizaci v AdS/CFT korespondenci. Nakonec představujeme vybrané teorie emergentní gravitace, jako je Jacobsonovo termodynamické odvození Einsteinových rovnic a Verlindeho entropická gravitace. Rovněž zkoumáme roli zobecněných entropií v emergentní gravitaci a to, jak mohou modifikovat Friedmannovu rovnici.

Klíčová slova: emergentní gravitace, termodynamika černých děr, Hawkingovo záření, zobecněný druhý zákon, holografický princip, entropické meze, Boussoova mez, AdS/CFT korespondence, Jacobsonův přístup, Verlindeho entropická gravitace, zobecněné entropie, modifikace Friedmanovy rovnice

Title:

Emergent gravity

Author: Jan Masák

Abstract: This work reviews the idea that gravity may be an emergent phenomenon rather than a fundamental interaction. We discuss the motivations behind this perspective, as well as the frameworks in which it has been developed. The review begins with black hole thermodynamics, including the four laws, Hawking radiation, and the generalized second law. Next, we turn to the holographic principle, focusing on entropy bounds—particularly Bousso's bound—and discussing its realization in the AdS/CFT correspondence. Finally, we present selected theories of emergent gravity, such as Jacobson's thermodynamic derivation of the Einstein equations and Verlinde's entropic gravity. We also examine the role of generalized entropies in emergent gravity and how they can modify the Friedmann equation.

Key words: emergent gravity, black hole thermodynamics, Hawking radiation, generalized second law, holographic principle, entropy bounds, Bousso bound, AdS/CFT correspondence, Jacobson approach, Verlinde entropic gravity, generalized entropies, modified Friedmann equation

Contents

Introduction

One of the central open problems in modern theoretical physics is the formulation of a consistent theory of quantum gravity. There are several motivations for seeking a quantum description of gravity. First, all other known fundamental interactions are successfully described in the framework of quantum theory, and (at present) there is no compelling reason to regard gravity as an exception. It is therefore natural to expect that a complete theory of nature will also be quantum in nature. This is also indirectly supported by various theoretical arguments for the necessity of quantizing gravity. Examples include, e.g., the incompatibility between classical gravity and the quantum description of matter fields in semiclassical gravity, which leads to conceptual inconsistencies such as violations of the uncertainty principle or superluminal signalling [1, 2], or Feynman's thought experiment involving the quantum superposition of massive objects and the consequent requirement that the gravitational field itself must be quantized to preserve the consistency of quantum mechanics [3]. In addition, it is difficult to envisage a framework in which a fully quantum world could emerge from a theory that is, even in part, of purely classical origin, since such scenarios are strongly constrained by Bell's theorem [4].

Quantizing gravity directly is, however, a notoriously difficult problem. From the point of view of quantum field theory, general relativity is famously non-renormalizable, even though it works remarkably well at low energies when treated as an effective quantum field theory [5]. The two most studied candidates for quantum gravity are loop quantum gravity and string theory. Loop quantum gravity faces difficulties in embedding itself consistently into a full theory of everything, while string theory has trouble making clear experimental predictions and suffers from challenges in constructing stable de Sitter vacua [6].

From a field theory perspective, one can ask whether massless spin-2 particles (gravitons) could arise as bound states in a theory of lower-spin fields. Weinberg [7] shows that if a consistent theory contains a massless spin-2 particle, its universal low-energy interactions must coincide with those of general relativity. However, the possibility of obtaining such a particle is excluded by powerful results of Weinberg and Witten [8]:

Theorem 0.1 (First spin 2 no-go theorem)**.** *A theory that allows the construction of a Lorentz-covariant conserved 4-vector current J_μ cannot contain massless particles of spin > 1 with non-vanishing values of the conceived charge $Q = \int J_0 \, d^3x$.*

Theorem 0.2 (Second spin 2 no-go theorem)**.** *A theory that allows a conserved Lorentz-covariant stress tensor $T_{\mu\nu}$ cannot contain massless particles of spin > 1.*

If gravity is emergent, these obstacles could be avoided, just as one does not need to quantize sound waves in order to understand them at the macroscopic level. The first idea of gravity as an emergent phenomenon is often traced back to Sakharov's induced gravity [9], where gravity emerges from quantum field theory in a way analogous to how hydrodynamics or continuum elasticity emerge from molecular physics [10].

There are also striking similarities between general relativity and thermodynamics. The laws of black hole mechanics [11] take a form analogous to the laws of thermodynamics, with black hole surface gravity playing the role of temperature and horizon area playing the role of entropy. This analogy becomes sharper through Hawking's discovery of black hole radiation. Another related analogy is with hydrodynamics. The membrane paradigm [12] describes black hole horizons as if they were dynamical membranes with physical properties such as viscosity and conductivity. It was developed before the AdS/CFT correspondence but later found an explicit realization through holography, much like black hole thermodynamics. This parallel evolution of black hole thermodynamics and the membrane paradigm toward explicit holographic realizations provides additional support for the idea that gravity may be emergent.

Further motivation for emergent gravity comes from the changing perspective on the nature of spacetime in modern theoretical physics [13]. In string theory, dualities such as T-duality, S-duality, and U-duality imply that the notion of a unique background spacetime can be ambiguous. Many approaches to quantum gravity suggest the existence of a minimal length scale, with attempts to probe smaller distances failing or leading to the creation of extended objects or black holes [14, 15]. It has also been suggested that general covariance might be a derived property rather than a fundamental one. The emergence of time is more speculative, with no fully worked-out examples, but there are strong conceptual reasons to consider it. For an attempt at describing the emergence of time see for example [16].

A particularly important example of emergent gravity is the AdS/CFT correspondence. Unlike many other ideas in this area, it provides an exact mathematical framework relating a theory of quantum gravity in anti-de Sitter space to a conformal field theory without gravity. This duality has inspired entire textbooks [17, 18] and has a wide range of applications, from condensed matter physics (AdS/CMT) to quantum chromodynamics (AdS/QCD). It also serves as a kind of experimental testing ground for ideas about quantum gravity, since calculations can often be carried out on one side of the duality and compared to phenomena on the other.

The final chapter of this work is devoted to theories of emergent gravity. Among these, Jacobson's theory [19] is perhaps the best-defined and most canonical. It derives Einstein's equations from thermodynamic considerations applied to local Rindler horizons, taking as input the proportionality between entropy and horizon area, and the Unruh temperature experienced by accelerated observers. Verlinde's theory [20], on the other hand, is a more recent proposal that approaches gravity as an entropic force, based on for example the famous Bekenstein's entropy bound [21]. While somewhat less precisely defined, it connects naturally to frameworks like AdS/CFT and has inspired numerous applications. We also discuss the role of generalized entropies in emergent gravity and how their use can lead, for example, to deformations of the Friedmann equations [22].

The structure of this work is as follows. In Chapter 1, we review black hole thermodynamics, which provides one of the earliest hints that gravity may be an emergent phenomenon. Chapter 2 introduces the holographic principle and its realization in the AdS/CFT correspondence, along with applications such as holographic hydrodynamics. Finally, Chapter 3 presents selected theories of emergent gravity, including those of Jacobson and Verlinde, and examines the possible applications of generalized entropy functionals in emergent gravity. Appendix A then contains useful definitions and concepts of general relativity used extensively throughout.

Conventions

Throughout this work, the following conventions are used, if not stated otherwise.

- We use the Planck units, the definition of which can be found in the Table 1. Newton's constant G, Planck's constant $\hbar$, the speed of light c, and Boltzmann's constant k_B all take in these units the numerical value 1.
- Einstein summation convention is used on all types of indices.
- Spatial vectors are typically represented in bold font, while spacetime vectors are distinguished by the use of indices.
- The abstract index notation is used as described in [23].
- If we say that some manifold (M, g_{ab}) is a spacetime, we mean that g_{ab} has signature $(+, -, -, -)$.

	Expression	Approximate value (in SI units)
Planck length (ℓ_p)	$\sqrt{\frac{\hbar G}{c^3}}$	$1.6 \times 10^{-35}\,\mathrm{m}$
Planck time (t_p)	$\sqrt{\frac{\hbar G}{c^5}}$	$5.4 \times 10^{-44}\,\mathrm{s}$
Planck mass (m_p)	$\sqrt{\frac{\hbar c}{G}}$	$2.2 \times 10^{-8}\,\mathrm{kg}$
Planck temperature (T_p)	$\sqrt{\frac{\hbar c^5}{G k_B^2}}$	$1.4 \times 10^{32}\,\mathrm{K}$

Table 1: Definition of Planck units in terms of universal constants and their approximate value in SI units.

Chapter 1

Black hole thermodynamics

1.1 Four laws of black hole dynamics

In the 1970s, physicists began to notice some interesting similarities between black hole physics and thermodynamics. The early study of this analogy is summarized in the four laws of black hole dynamics [11]. These are analogues of our beloved four laws of thermodynamics, which can be thought of as emerging from statistical physics. In contrast, we will see that the laws of black hole dynamics are theorems of pure general relativity and differential geometry.

The reader is encouraged to take a look at Appendix A for some concepts from general relativity that are used extensively in this chapter. Before discussing the four laws, we should also mention an important related theorem that is central to black hole thermodynamics in general. It is actually a group of theorems called black hole uniqueness theorems, formulated by Israel, Carter, Robinson, Mazur, Bunting and others. However, we will only give a non-rigorous summary of these theorems, which can be found in [24].

Theorem 1.1 (no-hair)**.** *Any isolated, stationary black hole that is a solution of the electrovacuum Einstein-Maxwell field equations in an asymptotically flat spacetime that contains no naked (not inside an event horizon) singularities and no closed time-like curves anywhere except possibly below the horizon is necessarily of the Kerr-Newman type.*

Note that axisymmetry is a consequence of this theorem, not an assumption. In official jargon, these theorems are sometimes called the 'no-hair' theorems because they imply that one needs very little information to describe a generic black hole. In fact, the Kerr-Newman metric is characterized by just three parameters (M, Q, J), where M is the mass of the black hole, Q is its charge, and J is its angular momentum (although a often appears in the metric instead of J, defined as $a = \frac{J}{M}$). This is in stark contrast to the stars from which these black holes often form. A large amount of information, proportional to its entropy, is needed to accurately describe a star. Therefore, as the star collapses, it appears to 'lose its hair'. This loss of information is one of the reasons why we ascribe entropy to black holes.

It is certainly not obvious that gravitational collapse settles into stationary spacetime. However, one can argue in favour of it, for example on the basis of the so-called Price's Law [25], which very roughly amounts to 'Whatever can be radiated will be radiated.' This means that all non-stationary features of gravitational and electromagnetic fields are radiated away. If we relate thermodynamic equilibrium to stationarity, it is similar to thermodynamic systems that tend towards thermal equilibrium. This analogy between equilibrium and stationarity is further supported by the fact that the zeroth and first laws of black hole dynamics hold for stationary spacetimes, as do their thermodynamic equivalents for equilibria.

Let us now finally move to the promised four laws of black hole dynamics.

0. law of black hole dynamics

In the formulation of this law of thermodynamics, we draw from [26]. Similarly, as in rigidity theorems found in Appendix A, we have two versions with different assumptions. The first is due to Carter.

Theorem 1.2 (0. law - Carter)**.** *Let us have an asymptotically flat spacetime that is, additionally, static or stationary, and axisymmetric with the t-ϕ orthogonality property. Then, for any black hole in this spacetime, the surface gravity remains constant over its event horizon.*

Note that in Appendix A we define surface gravity only for Killing horizons, not event horizons. However, under these assumptions, rigidity theorems hold, and every event horizon is also a Killing horizon.

The second formulation is by Carter, Bardeen, and Hawking, and it reads.

Theorem 1.3 (0. law - Carter, Bardeen, Hawking)**.** *In a spacetime where the Einstein equations, together with the dominant energy condition, are satisfied, the surface gravity remains constant over any Killing horizon.*

The analogy with the zeroth law of thermodynamics should be clear. As already mentioned, the stationarity of a black hole corresponds to a system being in thermal equilibrium. Surface gravity is then the analogue of temperature. One might argue that in Theorem 1.3, we do not explicitly mention stationarity. However, to apply the theorem to event horizons, we need the rigidity theorem, where stationarity is essential.

Correspondence with thermodynamics is in this case a looser one [27]. This is because the constancy of temperature in a system in thermal equilibrium is a consequence of the zeroth law rather than the zeroth law itself. This law instead posits that the relation of thermal equilibrium is an equivalence relation on pairs of thermodynamic systems. Also, there are many reasons to believe that the black hole temperature should be in pure general relativity absolute zero [11]. Perhaps the easiest way to see this is from the fact that black holes are perfect absorbers. However, when quantum fields come into play the analogy becomes stronger as we explain in Section 1.2.

1. law of black hole dynamics

Let us again further loosen up the rigour and provide an approximate formulation of the first law. Here we used [24, 26].

Theorem 1.4 (1. law)**.** *Let us have an asymptotically flat stationary spacetime which is a solution to the electrovacuum Einstein-Maxwell equations and which contains a black hole $\mathcal{B}$. Then to first order, stationary variations of $\mathcal{B}$ satisfy in electrovacuum*

$$\delta M = \frac{\kappa}{8\pi}\delta A + \omega\delta J + \varphi\delta Q\,, \tag{1.1}$$

where M is black hole's mass, J is its angular momentum, Q is its charge, A is the proper area of $\partial\mathcal{B}$, ω is the parameter appearing in Theorem A.11 and $\varphi = A_a(p)(t^a(p) + \omega\phi^a(p))$ for some $p \in \partial\mathcal{B}$, A_a being an electromagnetic four-potential.

In this ω can be interpreted as the angular velocity of the event horizon. Specifically, this theorem can be easily shown to hold for the Kerr-Newman solution [24] which is often sufficient because the assumptions of the no-hair theorem 1.1 are satisfied. Conversely, under certain conditions, the assumptions can be relaxed, for example, to include non-stationary perturbations and modifications of the Einstein-Hilbert action [26].

If we compare this to the first law of thermodynamics, we see that we can view M as the internal energy, and $\frac{\kappa}{8\pi}\delta A$ should correspond to TdS. This is because, from the zeroth law, we already know that $\kappa \propto T$, and from the second law, we will see that $A \propto S$. We can now integrate the relation $\frac{\kappa}{8\pi}dA = TdS$ to obtain

$$S = \frac{\kappa A}{8\pi T}\,, \tag{1.2}$$

where we set the integration constant to 0 because a black hole with no area should have no entropy. This equation will be useful in Section 1.3.

2. law of black hole dynamics

Paraphrasing a theorem that can be found in [28], we give a rigorous formulation of this law below.

Theorem 1.5 (2. law). *Let (M, g_{ab}) be a regular predictable spacetime developing according to Einstein equations from partial Cauchy surface $\Sigma(\tau) \subset M$, at which there is no cosmological constant ($\Lambda = 0$) and the null energy condition holds. If we then have $n \in \mathbb{N}$ black holes $\{\mathcal{B}_i(\tau)\}_{i\in\hat{n}} \subset \Sigma(\tau)$ at time τ, a black hole $\mathcal{B}_1(\tau') \subset \Sigma(\tau')$ at a later time τ' and $\forall i \in \hat{n}$: $J^+(\mathcal{B}_i(\tau)) \cap \mathcal{B}_1(\tau') \neq \emptyset$, then it holds that*

$$A(\partial\mathcal{B}_1(\tau')) \geq \sum_{i\in\hat{n}} A(\partial\mathcal{B}_i(\tau))\,,$$

where $A(\partial\mathcal{B}_i(\tau))$ is proper area of the event horizon of $\mathcal{B}_i(\tau)$ and the equality can be realized just for $n = 1$.

We will omit (even in Appendix A) the definition of regular predictable spacetime, as it is rather complex and can be found in [28]. Perhaps the most important element of this assumption is the absence of naked singularities and closed timelike curves in the spacetime.

It is worth noting that the analogue of Theorem 1.5 is stronger than the second law of thermodynamics. This is because the theorem holds even for $n = 1$, meaning the event horizon area of any single black hole cannot decrease. In contrast, the entropy of a thermodynamic system can decrease if entropy is transferred to another system in a way that ensures the total entropy does not decrease. However, this scenario is not possible for black holes, as they cannot bifurcate [28].

While Theorem 1.5 is true as a statement within pure general relativity, the situation changes again in the presence of quantum fields, as discussed in Section 1.2.

3. law of black hole dynamics

The last law of black hole dynamics is often omitted in the literature and, as we shall see, this has probably its reasons. One reason might be that it has a rather different status from the other laws, because, as of today, it has not been precisely formulated or proven, at least in a way that would be universally accepted.

It was first introduced together with the other laws in [11] in the simple form: *'It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.'* and this rough formulation might be the one thing people actually agree on. In [11] you can also find some arguments in favour of this conjecture.

One gets the corresponding statement in thermodynamics just by identifying $\kappa \leftrightarrow T$. But this form of the third law of thermodynamics is probably less common, and one more often encounters Nernst's formulation which basically posits that $S \xrightarrow{T\to 0} 0$. Alas, here we see another failure of the black hole dynamics-thermodynamics analogy because clearly extremal black holes have $\kappa = 0$ but $A \neq 0$, thus in black hole dynamics the analog of that formulation fails.

However, even here one finds under closer inspection that the correspondence might be saved. This is because Nernst's version of the third law has been found to fail even in thermodynamics for some systems [29]. In [30] one can find proof of Nernst's theorem using statistical physics. But in this proof, assumptions of ground state non-degeneracy and non-vanishing gap are found to play a key role, so in the absence of these assumptions one can sometimes find the theorem to be violated. In Section 2.3, we discuss a setup where a violation of the Nernst version of the third law in a thermodynamical system is dual, via AdS/CFT, to the violation in black hole dynamics.

A paper by Israel [31] claims to have formulated and proven the third law of black hole dynamics, but it seems that it is not universally accepted. Here is the formulation found in that paper:

Theorem 1.6 (3. law)**.** *Let us have a strongly future asymptotically predictable black hole spacetime. Then, in a continuous process in which some set of partial Cauchy surfaces $\Sigma(\tau)$ contains trapped surfaces for times $\tau < \tau_0$ but not for times $\tau > \tau_0$ must be the weak energy condition violated in a neighbourhood of the apparent horizon in $\Sigma(\tau_0)$.*

As with regular predictability, we omit the definition of strongly future asymptotically predictable spacetime, which can be found for example in [28] or [32] and is similar in spirit to regular predictability. However, the definition here is slightly modified in a way that is noted in [31], where one can also find the exact meaning of the phrase 'continuous process'. This formulation differs from that of previous laws in that it is formulated in terms of trapped surfaces and an apparent horizon rather than an event horizon. This may not be ideal, as will be discussed later.

There has been an interesting recent development regarding the third law. In papers [33, 34], the authors claim to have found counterexamples to both of the formulations of the third law of black hole dynamics presented. These papers are rather technical and have not yet passed peer review, so there remain good reasons for scepticism. The scenario considered by the authors is that of a charged scalar field undergoing a gravitational, spherically symmetric, collapse resulting in an extremal Reissner-Nordström black hole. In my opinion, it is quite possible this is indeed a counterexample to Israel's formulation. However, I think that there remains room for interpretation in the formulation of [11]. For example, I would regard this collapse as a highly non-equilibrium process which should not be counted as a 'step' in that formulation. Specifically, the gravitational analogue of thermodynamic adiabatic accessibility may well remain valid: there might still be no process that starts with a black hole and transforms it into an extremal one through steps during which the spacetime remains approximately stationary.

You may have noticed that in formulating the laws of black hole dynamics we have primarily used global notions such as the event horizon, asymptotic flatness or spacetime symmetries. However, we would prefer to formulate the four laws locally as their thermodynamic counterparts. This is one of the problems that suggest that this analogy may not be as strong as it is usually presented [27].

A major role, for example, plays the notion of an event horizon which is locally notoriously problematic. Maybe the best alternative for the event horizon is the apparent horizon, closely connected to trapped surfaces (both defined in Appendix A). These are used, for example, in Israel's formulation of the third law. There is a conjecture, referred to as weak cosmic censorship, which we will not state due to its technicality, however if true, it implies that all trapped surfaces, along with the apparent horizon, always lie within or on the event horizon. There even exist analogues of the laws of black hole dynamics using trapped surfaces [35]. However, trapped surfaces have their issues [27]. Perhaps the most significant of these problems is that trapped surfaces are dependent on the chosen foliation. There are even black hole spacetimes that can be foliated so that they contain no trapped surfaces. We have therefore stuck to the usual formulation of the laws of black hole dynamics in terms of event horizons.

As we have seen, the analogy with thermodynamics is problematic in many ways. On the other hand, it may hold up better than it has any right to if it is just a coincidence. Initially, physicists generally found the four laws a convenient analogy. However, Jacob Bekenstein in particular thought otherwise, based on his belief that black holes should have entropy. In the following subsections, we discover that the analogy is stronger than it might have seemed.

1.2 Hawking radiation

As we have already mentioned, many physicists, including Stephen Hawking, were opposed to Bekenstein's belief that black hole thermodynamics is more than just an analogy. Surely a black hole cannot have a temperature, they thought, because it would have to emit black-body radiation. Legend has it that in an attempt to prove Bekenstein wrong and show that this could not be the case, Hawking set out to calculate what would happen according to quantum field theory.

In his 1975 paper [36], Hawking considered a spacetime with a massless scalar field in which a black hole has formed by gravitational collapse. He used the semi-classical approximation of quantum field theory on a curved background. For such a calculation, one would arguably need a full quantum theory of gravity, but if we are not close to the singularity, this approximation should be sufficient. However, we will take a different route from Hawking's and follow primarily [37]. Our discussion will be quite mathematical, but a shorter, more intuitive one can be found for example in [15]. It turns out that what is important is the existence of an event horizon or, more generally, a Killing horizon, and this is what we will focus on.

Let us have a Schwarzschild black hole spacetime with the famous metric

$$g = \left(1 - \frac{r_s}{r}\right) dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \,, \tag{1.3}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$ is the metric of $\mathbb{S}^2$ and $r_s = 2M$ is the Schwarzschild radius. First, we need to calculate the surface gravity of the event horizon ($r = r_s$) for future use. This event horizon is in accord with Theorem A.11 also a Killing horizon of a Killing vector field which has in Schwarzschild coordinates components $\xi^\mu = \delta^\mu_t$. All we have to do is apply the formula (A.10). We have

$$\nabla_\mu \xi^\nu = \partial_\mu \xi^\nu + \xi^\rho \Gamma^\nu_{\rho\mu} = \Gamma^\nu_{t\mu} \,. \tag{1.4}$$

The only corresponding non-zero components of the affine connection are

$$\Gamma^t_{tr} = \frac{1}{2} g^{tt} \partial_r g_{tt} \quad \text{and} \qquad \Gamma^r_{tt} = -\frac{1}{2} g^{rr} \partial_r g_{tt} \,. \tag{1.5}$$

It follows that

$$\begin{aligned} g_{\nu\sigma} g^{\mu\rho} \nabla_\mu \xi^\nu \nabla_\rho \xi^\sigma &= g_{tt} g^{rr} (\Gamma^t_{tr})^2 + g_{rr} g^{tt} (\Gamma^r_{tt})^2 = \\ &= \frac{1}{4} g_{tt} g^{rr} (g^{tt})^2 \left(\frac{r_s}{r^2}\right)^2 + \frac{1}{4} g_{rr} g^{tt} (g^{rr})^2 \left(\frac{r_s}{r^2}\right)^2 = \frac{1}{2} \left(\frac{r_s}{r^2}\right)^2 \,, \end{aligned} \tag{1.6}$$

where we have used (1.4), (1.5) and $g_{tt} = -g^{-1}_{rr}$. If we substitute this into (A.10) and evaluate at $r = r_s$, we get

$$\kappa = \frac{1}{2r_s} = \frac{1}{4M} \,, \tag{1.7}$$

which is incidentally exactly the Newtonian value $\frac{GM}{r_s^2}$.

We now look at the metric in the Kruskal-Szekeres coordinates, where is the event horizon no longer singular.

$$g = \frac{32M^3}{r} e^{-\frac{r}{r_s}} (dV^2 - dU^2) - r^2 d\Omega^2 \tag{1.8}$$

Above the event horizon ($r \geq r_s$) are these coordinates given in terms of the Schwarzschild coordinates as

$$\begin{aligned} V &= \sqrt{\left(\frac{r}{r_s} - 1\right) e^{\frac{r}{r_s}}} \sinh \kappa t \,, \\ U &= \sqrt{\left(\frac{r}{r_s} - 1\right) e^{\frac{r}{r_s}}} \cosh \kappa t \,. \end{aligned} \tag{1.9}$$

If we now rescale the Kruskal-Szekeres coordinates (1.9) as

$$\begin{aligned} T &:= \frac{V}{\kappa\sqrt{e}} = \frac{1}{\kappa} \sqrt{\left(\frac{r}{r_s} - 1\right) e^{\frac{r}{r_s} - 1}} \sinh \kappa t \,, \\ X &:= \frac{U}{\kappa\sqrt{e}} = \frac{1}{\kappa} \sqrt{\left(\frac{r}{r_s} - 1\right) e^{\frac{r}{r_s} - 1}} \cosh \kappa t \,, \end{aligned} \tag{1.10}$$

we can rewrite the metric (1.8) into a better form

$$g = \frac{r_s}{r} \exp\left(1 - \frac{r}{r_s}\right) (dT^2 - dX^2) - r^2 d\Omega^2 \,. \tag{1.11}$$

From this, we see (using for example the Taylor series) that at some right neighbourhood of the event horizon ($r \in (r_s, r_s + \varepsilon)$), this metric can be approximated by

$$g \approx g_{\mathbb{M}^2} - r_s^2 d\Omega^2 \quad \text{where} \quad g_{\mathbb{M}^2} = dT^2 - dX^2 \,. \tag{1.12}$$

But this is just metric of $\mathbb{M}^2 \times \mathbb{S}^2(r_s)$ with $\mathbb{M}^2 = (\mathbb{R}^2, g_{\mathbb{M}^2})$ being the 2 dimensional Minkowski space and $\mathbb{S}^2(r_s)$ is a sphere of radius r_s. Further, we define for $r \geq r_s$

$$\rho := \frac{1}{\kappa}\sqrt{\left(\frac{r}{r_s} - 1\right) e^{\frac{r}{r_s}-1}} \tag{1.13}$$

and

$$\ell(r) := \int_{r_s}^{r} \sqrt{-g_{rr}(r')}\, dr' = \int_{r_s}^{r} \frac{dr'}{\sqrt{1-\frac{r_s}{r'}}} = r\sqrt{1-\frac{r_s}{r}} + r_s \text{arctanh}\left(\sqrt{1-\frac{r_s}{r}}\right) , \tag{1.14}$$

where $\ell(r)$ is the proper distance from the event horizon. Let also for future use establish an equivalence relation between two functions of the Schwarzschild radius as $f(r) \sim g(r) \iff \frac{f(r)}{g(r)} \xrightarrow{r \to r_s^+} 1$. It simply relates functions that behave the same in some small neighbourhood above the horizon. We then observe that $\rho(r) \sim \ell(r)$, so in the vicinity of the event horizon has ρ the meaning of proper distance from it. If we compare (1.13) and definition of the new coordinates (1.10), we see that

$$T = \rho \sinh \eta \quad \text{and} \quad X = \rho \cosh \eta \,, \tag{1.15}$$

where we introduced the coordinate $\eta := \kappa t$. But (1.15) is very similar to the parameterization of the trajectory of a Rindler observer (an observer moving through Minkowski space with constant four-acceleration), which is given in Minkowski coordinates by

$$x^\mu(\tau) = \left(\frac{1}{\alpha}\sinh \alpha\tau, \frac{1}{\alpha}\cosh \alpha\tau\right) . \tag{1.16}$$

Here τ is the proper time of the observer and α is the magnitude of four-acceleration which can be quickly verified by the following calculation.

$$a = \sqrt{-a^\mu a_\mu} = \sqrt{-\eta_{\mu\nu}\frac{d^2x^\mu}{d\tau^2}\frac{d^2x^\nu}{d\tau^2}} = \sqrt{\alpha^2\cosh^2\alpha\tau - \alpha^2\sinh^2\alpha\tau} = \alpha \,. \tag{1.17}$$

Looking at the parameterization (1.16), it is also clear that such an observer would follow a hyperbolic trajectory in Minkowski coordinates.

From Figure 1.1 we can observe that static observers ($r =$ const.) also follow hyperbolas in the Kruskal diagram. This is no coincidence. Indeed, we can calculate the magnitude of four-acceleration of these observers. Static observers move on integral curves of the Killing vector field ξ^a and therefore have four-velocity $u^a = \frac{\xi^a}{\xi}$ where in the Schwarzschild coordinates $\xi = \sqrt{g_{\mu\nu}\xi^\mu\xi^\nu} = \sqrt{g_{tt}}$. The formula for four-acceleration can be written as $a^a = u^c\nabla_c u^a$, and therefore we can calculate a using (1.4) and (1.5)

$$\begin{aligned} a^\mu a_\mu = g_{\mu\nu}\frac{\xi^\rho}{\xi}\nabla_\rho\frac{\xi^\mu}{\xi}\frac{\xi^\sigma}{\xi}\nabla_\sigma\frac{\xi^\nu}{\xi} = \frac{1}{\xi^4}g_{\mu\nu}\nabla_t\xi^\mu\nabla_t\xi^\nu = \frac{1}{\xi^4}g_{rr}(\Gamma^r_{tt})^2 = \\ = \frac{1}{4\xi^4}g_{rr}(g^{rr})^2(\partial_r g_{tt})^2 = -\frac{1}{4\xi^4}g_{tt}\frac{r_s^2}{r^4} \,, \end{aligned} \tag{1.18}$$

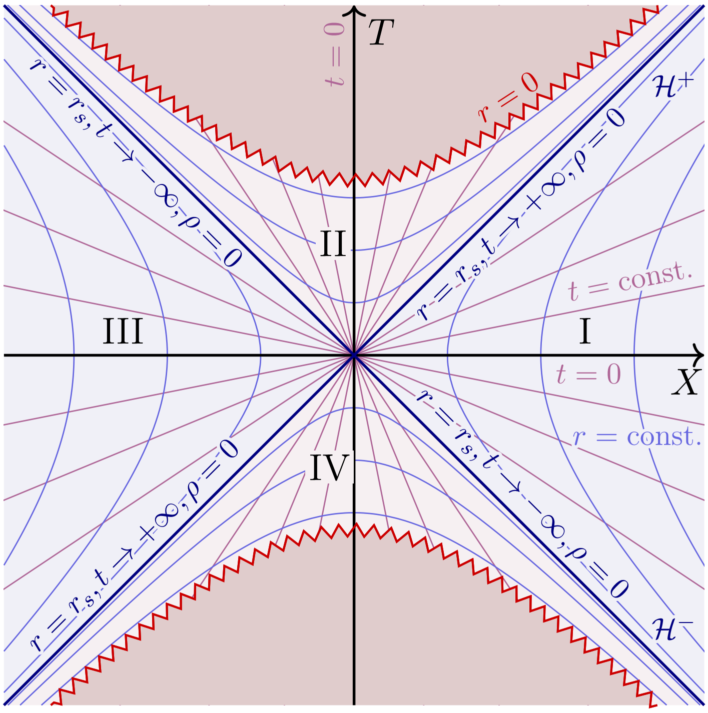

Figure 1.1: Kruskal diagram with the four patches representing the maximally extended Schwarzschild spacetime for constant values of θ and ϕ in coordinates (1.10). Each point can therefore be thought of as representing a 2-dimensional sphere. Regions III and IV are artefacts of the time reversibility of Einstein's equations and do not exist in a realistic stellar collapse. Hyperbolas of constant Schwarzchild time t are shown in purple and hyperbolas of constant r are shown in blue. $\mathcal{H}^+$ is called the future event horizon, $\mathcal{H}^-$ the past event horizon and $r=0$ is the singularity.

where we also again used $g^{rr} = -g_{tt}$. All in all, we have

$$a(r) = \frac{1}{2}\sqrt{\frac{r_s}{r^3\left(\frac{r}{r_s}-1\right)}}\,. \tag{1.19}$$

This shows that static observers have constant four-acceleration.

We can make the analogy between (1.15) and (1.16) even more vivid by making two more observations. One would be that $\rho \sim \frac{1}{a}$. The second observation can be made if one remembers that for our static observers in Schwarzschild coordinates $u^t = \frac{dt}{d\tau} = \frac{\xi^t}{\xi} = \frac{1}{\xi}$, therefore $t = \frac{\tau}{\xi}$ and $\eta = \kappa t = \frac{\kappa}{\xi}\tau$. The correspondence of (1.15) and (1.16) would be finished if we could near the horizon identify a and $\frac{\kappa}{\xi}$, but indeed $a(r) \sim \frac{\kappa}{\xi(r)}$.

The correspondence between Minkowski space with accelerated observers and Schwarzschild spacetime with static observers near the horizon is further apparent from Figure 1.1. If we focus on a small neighbourhood of the origin, where we are near the event horizon, it looks exactly like Minkowski spacetime (we could focus anywhere near the horizon, but then we would have shifted the origin) with coordinates (T, X, Y, Z), where Y and Z are constant. In Minkowski spacetime we also have Killing horizons, which are important for accelerated observers. These are horizons corresponding to the Killing vector field generating boosts $X\partial_T + T\partial_X$ passing through the origin. This Killing vector field can be written in the coordinates (ρ, η) defined in

(1.15) simply as ∂_η, which can be obtained from

$$\partial_\eta = \frac{\partial T}{\partial \eta}\partial_T + \frac{\partial X}{\partial \eta}\partial_X = \rho\cosh\eta\,\partial_T + \rho\sinh\eta\,\partial_X = X\partial_T + T\partial_X\,. \tag{1.20}$$

Such horizons are null hypersurfaces which, like the event horizons $\mathcal{H}^\pm$ in Figure 1.1, bound regions of spacetime inaccessible to the observers. The locations of these horizons also match exactly. It is important to note that ∂_η with $\eta = \kappa t$ is also the Killing vector field generating the Schwarzschild event horizons and that η is related to the proper time of the observers in both cases. This, together with the fact that this vector field vanishes at the Killing horizons, is in a sense what is essential for this effect, although it may not seem so from the following derivation.

Now that we established that in the radial direction r, a static observer near the event horizon can be thought of as a Rindler observer with acceleration $a = \frac{1}{\rho}$, we show that in both cases they should view the vacuum as a thermal state. For a Rindler observer, we take the metric of $\mathbb{M}^4$ in Minkowski coordinates

$$g_{\mathbb{M}^4} = g_{\mathbb{M}^2} - dY^2 - dZ^2 \tag{1.21}$$

and for the Schwarzschild case the metric (1.12). In both cases the relevant part is $g_{\mathbb{M}^2}$, on which the coordinate transformation (1.15) can be performed to obtain

$$g_{\mathbb{M}^2} = \rho^2 d\eta^2 - d\rho^2\,. \tag{1.22}$$

Here the definition of coordinates (η, ρ) differs in the two cases. For a Rindler observer $(\eta, \rho) = \left(\alpha\tau, \frac{1}{\alpha}\right)$ and for the Schwarzschild case were these already defined ($\eta = \kappa t$ and ρ was defined by (1.13)).

To show that the quantum field theory observed by these observers has a non-zero temperature, we use a Euclidean path integral. We go over to Euclidean signature by performing a Wick rotation $(t, \tau) \to (it, i\tau)$ resulting into $\eta \to \theta := i\eta$ which puts (1.22) into the form

$$-g_{\mathbb{M}^2} = \rho^2 d\theta^2 + d\rho^2\,. \tag{1.23}$$

The form of this metric is exactly minus that of flat metric in polar coordinates and therefore Riemann tensor must be everywhere 0. But following [38] we further inspect if parallel transport is really integrable.

Let us assume that the time dimension θ is periodic with a period ϕ so that $\theta \sim \theta + \phi$. We take a vector V_0^μ and parallel transport it around a closed curve $x^\mu(s) = (\theta(s), \rho(s)) = (s, \rho_0)$. To do that we solve the equation for parallel transport

$$\frac{dV^\mu}{ds} + V^\sigma \Gamma^\mu_{\sigma\nu}\frac{dx^\nu}{ds} = 0 \tag{1.24}$$

with the initial condition $V^\mu(0) = V_0^\mu$. In this case, the only non-zero Christoffel symbols are

$$\Gamma^\rho_{\theta\theta} = -\rho \quad \text{and} \quad \Gamma^\theta_{\rho\theta} = \Gamma^\theta_{\theta\rho} = \frac{1}{\rho}\,. \tag{1.25}$$

This means that (1.24) is in this case specifically

$$\begin{aligned} \dot{V}^\theta(s) + \frac{1}{\rho_0}V^\rho(s) = 0\,, \\ \dot{V}^\rho(s) - \rho_0 V^\theta(s) = 0\,, \end{aligned} \tag{1.26}$$

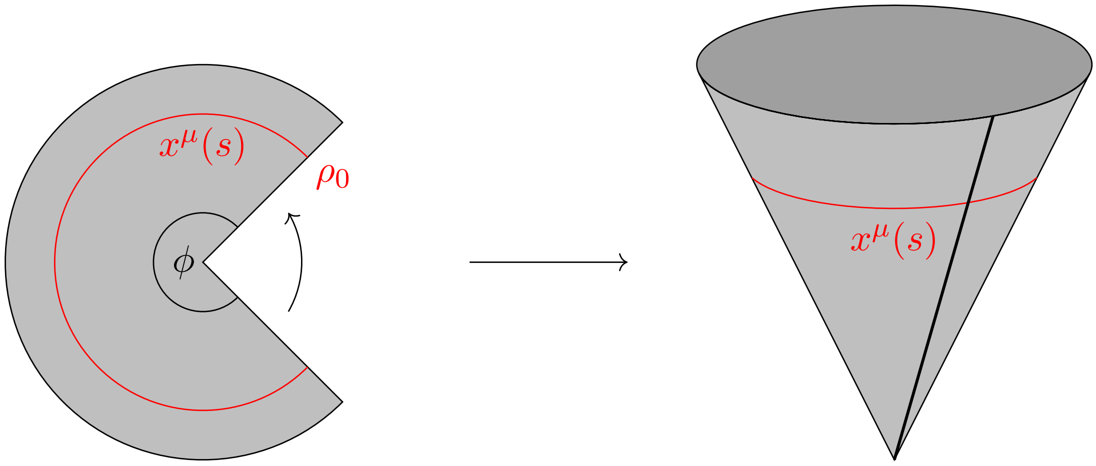

Figure 1.2: On the left side we see a circle with a circular cutout of angle $2\pi - \phi$. It contains a circular curve $x^\mu(s)$ on the radius ρ_0. The cone on the right is obtained by gluing the two sides of the cut together.

where we introduced $\dot{V}^\mu := \frac{dV^\mu}{ds}$. If we now take the derivative of the first equation and substitute the second, we get

$$\ddot{V}^\theta(s) + \dot{V}^\theta(s) = 0 \tag{1.27}$$

and by the same procedure, just swapping the roles of the first and second equations, we would get the same equation for ρ. So both equations can be written in a concise way as

$$\ddot{V}^\mu(s) + \dot{V}^\mu(s) = 0\,. \tag{1.28}$$

This has the general solution

$$V^\mu(s) = A^\mu \cos s + B^\mu \sin s\,, \tag{1.29}$$

where A^μ and B^μ are constant vectors. We have additionally initial conditions

$$V^\mu(0) = A^\mu = V_0^\mu \quad \text{and} \quad \dot{V}^\mu(0) = B^\mu = \begin{pmatrix} -\frac{V_0^\rho}{\rho_0} \\ \rho_0 V_0^\theta \end{pmatrix}\,. \tag{1.30}$$

Finally, we have the full solution

$$\begin{aligned} V^\theta(s) &= V_0^\theta \cos s - \frac{V_0^\rho}{\rho_0} \sin s\,, \\ V^\rho(s) &= V_0^\rho \cos s + \rho_0 V_0^\theta \sin s\,. \end{aligned} \tag{1.31}$$

Because sine and cosine are 2π periodic it must hold $\phi = 2\pi$ for parallel transport to be integrable. For a different period, the so-called conical singularity would occur at the origin. This can be understood intuitively from Figure 1.2. For example, if the period were smaller, it would correspond to cutting a circular section out of a plane and then gluing it together. This would produce an infinite cone with a singularity at its tip.

We see that in Euclidean signature θ is forced to be 2π periodic. But we know that the period of time dimension of a Euclidean quantum field theory at finite temperature is just $\beta = \frac{1}{T}$ where

T is the temperature[1]. For a Rindler observer with acceleration α we have $\theta = i\alpha\tau$, therefore the period of τ is $\frac{2\pi}{\alpha}$ and the temperature

$$T_U = \frac{\alpha}{2\pi} . \tag{1.32}$$

This effect where accelerated observers find themselves in a thermal bath of temperature T_U is called the Unruh effect [39].

Since we have shown that a static observer near the horizon in Schwarzschild spacetime is equivalent to a Rindler observer, the temperature seen by this observer should be the same. But we can still look at the period corresponding to Schwarzschild time, which is (as a result of $\theta = i\kappa\tau$)

$$T_H = \frac{\kappa}{2\pi} . \tag{1.33}$$

This differs from (1.32) exactly by the redshift factor, because Schwarzschild time is the time of observers at infinity, and they should see the temperature redshifted.

Our calculation applies only to a Schwarzschild black hole, but Hawking argued [36] that the formula (1.33) can be extended to rotating and charged black holes. The only difference would be in the form of surface gravity.

This approach to calculating the temperature (1.33) was quite different from Hawking's, but we got the same result. An advantage of this derivation is that we have shown that for these observers every quantum field must be in a thermal state. Hawking's derivation is done for a specific field and only concludes that the vacuum expectation value of particle number is that of thermal radiation. However, his calculation has the great advantage of being done for a real black hole formed by gravitational collapse, where we had assumed an eternal black hole. In conclusion, this discussion shows that the temperature discussed in Section 1.1 is in a sense very real, and we have also fixed its value.

It is interesting to note that a Schwarzschild black hole has a negative heat capacity [40]. This is because for a Schwarzschild black hole holds (1.7). And so the formula (1.33) becomes

$$T_S = \frac{1}{8\pi M} . \tag{1.34}$$

From the first law of black hole dynamics 1.4 we know that the black hole mass plays the role of the internal energy. This leads us to

$$C = \frac{\partial M}{\partial T_S} = -\frac{1}{8\pi T_S^2} , \tag{1.35}$$

where C is the heat capacity, and we do not need to specify the process because for a Schwarzschild black hole T is the only state variable. Using (1.34) we also see that $T_S = \frac{T_p m_p}{8\pi M_\odot}\frac{M_\odot}{M} \doteq 1.6 \times 10^{-7} \frac{M_\odot}{M} K$, so the temperature is very small for standard black holes.

[1] The thermal partition function of a quantum field theory at temperature T is given by $Z = \mathrm{Tr}\left(e^{-\beta H}\right)$ with $\beta = 1/T$ and H the Hamiltonian. This trace can be represented as a Euclidean path integral which for the example of a scalar field reads

$$\mathrm{Tr}\left(e^{-\beta H}\right) = \int_D \mathcal{D}\phi \, e^{-S_{\mathrm{euc}}(\phi)} ,$$

where $S_{\mathrm{euc}}(\phi) = \int_0^\beta d\tau \int d^{d-1}x \, \mathcal{L}(\phi, \partial\phi)\big|_{t=-i\tau}$, and the domain of integration is $D = \{ \phi \mid \phi(\tau+\beta, \boldsymbol{x}) = \phi(\tau, \boldsymbol{x}) \}$, that is, field configurations periodic in Euclidean time with period β (the imaginary time defined as $\tau = it$).

1.3 Generalized second law of thermodynamics

Let us now gather the insides of previous sections to introduce the generalized second law of thermodynamics. As with the four laws of black hole dynamics, we will mention just a few arguments in its favour but many more were put forward by Bekenstein, some for example in the paper [41]. He first presented the generalized second law in his 1972 paper [42], which contains two arguments against the standard formulation of the second law that we will focus on.

Already in Section 1.1 we commented on the apparent loss of information due to the no-hair theorem 1.1. From the point of view of pure general relativity, the entropy of black hole S_{BH} is exactly 0 since it is fully described by the three parameters (M, Q, J) and in some sense it has no microstates.

Jacob Bekenstein was a student of Archibald Wheeler, and in 1971 Wheeler came to Bekenstein with an interesting problem. Suppose a nefarious creature wanted to commit a crime against the second law of thermodynamics. Bekenstein called it Wheeler's demon [43]. If the creature takes a box full of radiation with some entropy and throws it into a black hole, the second law is violated as $S_{BH} = 0$. Of course, there are many reasons to believe that the inside of black holes is very different from the picture that general relativity provides. Perhaps the information about the object is still preserved just below the horizon, inaccessible to an outside observer. But from the outside, we would still have no way of checking whether the second law holds, so it would have the status of being observationally unverifiable. This suggests that we need a law that goes beyond the standard second law.

The second argument for abandoning the standard formulation was borrowed by Bekenstein from Geroch. Let us suppose that the demon continues with his devious plans and wants to construct a perpetual motion machine of the second kind. He connects his box full of thermal radiation to a rope, which he then attaches to an engine doing work. By lowering the box towards a Schwarzschild black hole, he converts the potential energy of the radiation into work. At the event horizon, all the energy mc^2 of the radiation is extracted. He dumps the radiation into the black hole and repeats the process. Thus, it appears that all the heat in the radiation has been converted into useful work, in violation of the second law of thermodynamics.

Of course, in 1972, Bekenstein did not know about Hawking radiation, but it provides yet another good argument in favour of the generalized second law. So far we have discussed the problems associated with the standard second law of thermodynamics, but the second law of black hole dynamics is also in trouble. This is because Hawking radiation provides a way of reducing the area of the event horizon. Obviously, Theorem 1.5 still holds as a mathematical theorem, but the assumptions are violated because the energy-momentum tensor in quantum field theory does not satisfy the null energy condition [26].

We have seen that both the second law of thermodynamics and the second law of black hole dynamics are in danger. Bekenstein suggested that we could save the situation by introducing the generalized second law of thermodynamics, which combines the two.

The first step towards this law is to take seriously the analogy between black hole entropy and the event horizon area. We have prepared (1.2) just for this occasion. From Hawking, we know that the formula for the temperature of a black hole should be (1.33). Substituting (1.33) into (1.2) we get

$$S_{BH} = \frac{A}{4}\,, \tag{1.36}$$

which is the famous Bekenstein-Hawking entropy. For a Schwarzschild black hole in standard units

$$S_{BH} = k_B \frac{A}{4\ell_p^2} = k_B \frac{4\pi r_s^2}{4\ell_p^2} \doteq 1.5 \times 10^{54} \left(\frac{M}{M_\odot}\right)^2 JK^{-1}, \tag{1.37}$$

which is an extremely large value. Indeed, in Section 2.1 we will see that there are reasons to believe that black holes have the largest entropy that can fit into a given region. It is also likely that most of the entropy of the universe is stored in black holes. It is perhaps interesting to note that both (1.34) and, for a Schwarzschild black hole, (1.36) contain, in standard units, all the fundamental constants, which may indicate that something profound is going on.

The generalized second law of thermodynamics then basically consists of placing the Bekenstein-Hawking entropy on the same footing as the entropy of matter to the extent that it enters the second law of thermodynamics. We can then simply formulate the generalized second law as

Generalized 2. law: *The total generalized entropy $\mathcal{S} := S_{BH} + S$ of the universe never decreases with time.*

Here S_{BH} is the Bekenstein-Hawking entropy of all black holes in our universe, calculated using the formula (1.36). S can then be defined as the von Neumann entropy of the quantum state of the outside of the black hole horizons, which reads

$$S = -\operatorname{Tr}(\rho_{\text{out}} \ln \rho_{\text{out}}) \tag{1.38}$$

Where the statistical operator ρ_{out} is obtained from the global quantum state ρ by tracing out the field degrees of freedom inside the black holes

$$\rho_{\text{out}} = \operatorname{Tr}_{\text{in}} \rho \,. \tag{1.39}$$

On the one hand, $\mathcal{S}$ is probably only a semi-classical concept. That is, if we assume that black hole evaporation is unitary, because then we suspect that the interior of a black hole should be entangled with its exterior. This is not captured in $\mathcal{S}$, as can be seen for example from the violation of the generalized second law after a Page time when both S and S_{BH} are decreasing [44]. On the other hand, it is interesting to note that $\mathcal{S}$ is cut-off independent [45, 46]. Both G (contained in S_{BH}) and S (which contains the divergent vacuum entanglement entropy) are sources of divergences and depend on the cut-off. However, when combined into $\mathcal{S}$, these divergences cancel, leaving a finite part that is invariant under the renormalization group flow. This suggests that $\mathcal{S}$ could provide us with information present in the full theory of quantum gravity.

This new law could solve all the problems presented. When an object enters a black hole, the area of the event horizon increases, and it could increase in exactly the way that ensures that $\mathcal{S}$ increases. However, this is far from obvious because the increase in horizon area does not seem to depend on the entropy of the object, only on its energy. One could imagine that objects with very high entropy and low energy could violate the generalized second law. The extreme case would be the Geroch process, where all the energy of the radiation is extracted, leaving only the entropy. However, as discussed in the subsection 2.1.1, it has been argued on several grounds that the generalized second law is impossible to violate by this process due to practical constraints. One solution has been to introduce an entropy bound, which is satisfied by systems in nature, and which limits the entropy of an object by its dimensions and energy. Furthermore, it has been checked [47] that the decrease in black hole area due to Hawking radiation is compensated by the entropy of the outgoing radiation, in accordance with the generalized second law.

Let us mention that there is another interesting argument attributed to Geroch as to why the Geroch process does not produce a perpetual motion machine of the second kind [43]. It has been discussed for example in [15], and it suggests that because in reality the box has some dimensions and no part of the box can cross the event horizon, the box cannot be lowered all the way to the event horizon and the efficiency is not 100%. Perhaps surprisingly, the formula for the efficiency of this process is the same as for the Carnot heat engine, if the temperature of the heat sink is set to a temperature very close to Hawking's temperature.

As with the standard second law of thermodynamics, the generalized second law is regarded as a statistical law and there is always a non-zero probability that it will be violated, but for a macroscopic system this probability is vanishingly small. Since we do not know the microscopic origin of Bekenstein-Hawking entropy, we have no way of deriving the generalized second law from statistical physics. Instead, it has been argued that the law should hold in a large number of thought experiments, each of which adds to its credibility [43]. There have been numerous cases where it seemed that the general second law should not hold, but in the end it was shown that it did, and it emerged stronger than before.

Chapter 2

Holographic principle

2.1 Entropy bounds

This section follows [40] if not stated otherwise. We have indicated in Section 1.3 that if we want the generalized second law of thermodynamics to hold, we need to postulate a bound on the entropy of matter. Another point of view would be that the generalized second law if taken as a law of nature implies some entropy bound. In the first two subsections 2.1.1, 2.1.2 we show how one can argue under certain assumptions for two different entropy bounds based on the generalized second law. At the end of Subsection 2.1.2 we then discuss the relationship between them.

Because these entropy bounds have their shortcomings, we also introduce the so-called Bousso's bound in Subsection 2.1.3. As we will see, it has many advantages and is probably the best formulated entropy bound currently available. We will also show how it relates to the two previously mentioned bounds, the generalized second law of thermodynamics and discuss its empirical status. All of this section is then a basis for the following Section 2.2, where we show how an entropy bound could lead one to postulate the holographic principle.

2.1.1 Bekenstein bound

The first one to propose an entropy bound was Jacob Bekenstein. This bound posits [48] that if the generalized second law of thermodynamics is to hold, the entropy of any weakly gravitating matter system in asymptotically flat space must satisfy

$$S \leq 2\pi ER = \frac{2\pi k_B}{\hbar c} ER \,. \tag{2.1}$$

Here E is the total energy of the matter system and R is the radius of the smallest circumscribing sphere at rest, both defined at the flat space at infinity. Notice that neither E nor R are covariant quantities, so we must assume that the spacetime is not significantly deformed. This is a major drawback of this entropy bound.

On the other hand, an advantage of this bound is that (2.1), despite its original derivation coming from black holes shown below, does not include G. This suggests that it could hold even in the absence of gravity, unlike the other entropy bounds we will see. Indeed, when properly formulated (excluding vacuum entropy), it is not hard to prove it in a relativistic quantum field theory, as has been done in [49]. It turns out that it is mainly protected by the uncertainty relations.

Let us now present an argument as to how the generalized second law applied to the Geroch process may imply the Bekenstein bound. Let us use a purely classical treatment in a regime of weak gravity and asymptotic flatness. We take a thermodynamic system of energy E and radius R (as defined above) and begin a Geroch process, already described in Section 1.3. That means lowering the box towards a Schwarzschild black hole as close as possible to extract as much energy as possible to tighten the entropy bound. Because we now consider that the system has some dimensions we cannot lower it all the way to the event horizon, but just about so the centre of mass of the system is about a proper distance R from the horizon (notice this assumption).

We would like to know what is now the energy of the system when length contraction and gravitational redshift are at work. The energy as measured by observer at asymptotic infinity is given by the projection of four-momentum onto the time Killing vector field $\xi^a = \partial_t$ as

$$\tilde{E} = p_a \xi^a = m u_a \xi^a = \frac{\xi_a \xi^a}{\xi} E = \xi E \,. \tag{2.2}$$

As in Section 1.2 we have used that static observers move on integral curves of ξ^a, therefore $u^a = \frac{\xi^a}{\xi}$ and we noticed that m is the energy the system would have if brought to rest at infinity E. If we also borrow from Section 1.2 the equivalence relation $\sim$ together with some other results, we can write $\frac{\xi(r)}{\kappa} \sim \frac{1}{a(r)} \sim \rho(r) \sim \ell(r)$ and because $\sim$ is indeed an equivalence relation, we can express the redshift factor in terms of proper distance from the event horizon as $\xi(r) \sim \kappa \ell(r)$. To apply this approximation we have to be near the horizon so that $R \ll r_s$, but we can always choose a black hole large enough so this holds. If we release the system when its centre of mass is about a proper distance R from the horizon the change of mass of the black hole can be estimated as

$$\delta M = \tilde{E}\Big|_{\ell=R} = \xi|_{\ell=R} E \simeq \kappa R E \,. \tag{2.3}$$

For a Schwarzschild black hole the first law of black hole dynamics (1.1) reduces to $\delta M = T_H \delta S_{BH}$ and because the Hawking temperature is given by (1.33), the surface gravity cancels leaving us with

$$S \leq \delta S_{BH} \simeq 2\pi E R \,. \tag{2.4}$$

The first inequality is where entered the generalized second law giving us $\delta S_{BH} - S \geq 0$.

In this derivation, we pointed out the additional assumption that you cannot get the centre of mass of the system closer to the horizon than the proper distance R. This would be the case for example for a spherically symmetric object. In contrast, let us consider a homogeneous box with one of the dimensions significantly smaller and label the length of this dimension h. If we were to repeat the previous argument for this box, but we oriented the shorter dimension in the radial direction, we would get inequality

$$S \lesssim 2\pi E \frac{h}{2} \,. \tag{2.5}$$

But this leads to an obvious contradiction, because for a given h we can always find lengths of the other dimensions so large that the volume would be the same as for a larger h and the box should be able to hold a similar amount of entropy (certainly not arbitrarily small amount).

But so far we treated the problem only classically and the quantum effects could be very important. Indeed, Unruh and Wald [26, 50] argued that quantum effects invalidate the derivation given and actually the Bekenstein bound is not needed for the validity of the generalized second law. They pointed out that even though Hawking radiation as viewed from infinity is negligible for standard-sized black holes from the point of view of a static observer near the horizon it can play a large role. We can get a glimpse into why this is so using the formula for Unruh radiation observed by Rindler observer with four-acceleration α (1.32). Because in Section 1.2 we established that static observers near the horizon with four-acceleration $a(r)$ are equivalent to Rindler observers with four acceleration $\alpha = a(r)$, substituting the formula (1.19) we get

$$T(r) = \frac{a(r)}{2\pi} = \frac{1}{4\pi}\sqrt{\frac{r_s}{r^3\left(\frac{r}{r_s}-1\right)}} \xrightarrow{r \to r_s^+} +\infty\,. \tag{2.6}$$

It has been argued that this Hawking radiation would provide a buoyancy force that would push against the object as it approaches the event horizon. The discussion regarding entropy would also be different, as the volume occupied by the system would be replaced by high entropy radiation after it falls into the black hole.

However, Bekenstein replied to these concerns and argued back that the Unruh radiation affects only the lowest layer of the system. This would mean that the argument based on the Geroch process should account for the Unruh radiation only in the case of a very flat system where we know the classical treatment fails anyway [21]. Hence, Bekenstein's argument for the necessity of the Bekenstein bound for the generalized second law would be valid. It is however still controversial whether the generalized second law implies the Bekenstein bound [40].

2.1.2 Spherical entropy bound

Perhaps a less controversial argument for an entropy bound is that for the so-called spherical bound. One of the first papers in which the idea of this heuristic derivation appeared was the article [51], but we will follow more the treatment of [26, 40]. As we will see, the derivation may be valid, but it is based on some rather strong assumptions, which are often violated even in realistic situations. Nevertheless, this bound is important mainly because it serves as a motivation for a better formulated bound discussed in the next Subsection 2.1.1.

Let us have a spacetime that allows black hole formation (e.g. asymptotically flat). In this spacetime we imagine an isolated system of energy E (not a black hole) and entropy S. We again consider the smallest sphere circumscribing the system and denote its area A. For this area to be well-defined, we assume that the system is spherically symmetric (hence the name of the bound) or weakly gravitating, and we also require that the area is approximately constant in time (the object is stable on an appropriate timescale).

Take M to be the mass of a black hole with horizon A. Suppose that we could collapse a spherical shell of mass $M - E$ onto the system, transforming it into a black hole. This is a reasonable assumption, because if the collapse is perfectly spherically symmetric and the shell manages to get inside the Schwarzschild radius, we know that the outside must be described by the Schwarzschild solution, and it is a black hole. If spherical symmetry is broken, we can still use the hoop conjecture, which states that if an object is compressed so that it can be enclosed in a sphere with its Schwarzschild radius, it will necessarily collapse into a black hole. This conjecture remains unproven, but there is much evidence for its validity (e.g. from computer simulations) [25].

If the shell initially has a positive entropy S_{shell}, then the generalized entropy evolves in the process as $\mathcal{S} = S + S_{\text{shell}} \to \mathcal{S}' = \frac{A}{4}$, and since according to the generalized second law of thermodynamics the generalized entropy cannot decrease, we are left with

$$S \leq S + S_{\text{shell}} \leq \frac{A}{4} = \frac{Ac^3}{4G\hbar}\,. \tag{2.7}$$

This is the sought-after spherical entropy bound. Note that it explicitly includes G, and therefore we cannot expect the bound to hold in the absence of gravity, unlike in the case of the Bekenstein bound. Also, since the upper bound in (2.7) is exactly the Bekenstein-Hawking entropy of the corresponding black hole, black holes are the most entropic objects that can fit into a given region, at least under appropriate assumptions. We have already pointed this out in Section 1.3, where we also calculated the value of the Bekenstein-Hawking entropy in standard units (1.37) and found that it is indeed a huge number. It turns out that in reality, it is very difficult to even approach the saturation of (2.7) with ordinary matter.

Now let us point out some hidden assumptions in the previous argument. For example, in [26], Wald points out that we assume that any sufficiently massive object will collapse into a black hole. It may be that for a system with entropy greater than $\frac{A}{4}$ the collapse would simply not occur to prevent violation of the generalized second law. However, it seems to us that this goes against the current state of the art in general relativity, because for a large system, the collapse should be well within the regime we understand, and quantum gravity is not needed, but it may be that the presence of such a high entropy has some exotic effects. Wald also argued that another hidden assumption is that the entropy of the whole region where the black hole resides is always given by the Bekenstein-Hawking entropy. For instance, one could imagine jumping into the black hole and exciting additional degrees of freedom. In this case, the argument would not hold because the system would already be a black hole.

Another objection to entropy bounds, in general, is the so-called species problem [40]. It is suggested that all entropy bounds can be violated if we make the number of species of fields sufficiently large. But how large a number would we need? For example, to violate the spherical entropy bound for systems with a spherical boundary of area up to A, the number would need to be about larger than A, which for a proton-sized object would be more than 10^{40}. Although it is not difficult to write down such a theory, it is ruled out by the experimental evidence. This is because it would make black holes larger than the Planck scale very unstable since the evaporation time of Hawking radiation is inversely proportional to the number of fields. This is actually why the reasoning of the spherical bound argument does not apply here, as the black hole would evaporate before it had a chance to form.

The relationship between the spherical and the Bekenstein bound is that the Bekenstein bound is stronger, if we assume the latter to hold even for spherically symmetric systems with strong gravity. If the system is to be gravitationally stable the radius R as defined in Subsection 2.1.1 must be larger than its Schwarzschild radius $r_s = 2M$ as one can argue for example based on the hoop conjecture mentioned earlier. We then have

$$S \leq 2\pi RE \leq 2\pi RM \leq \pi R^2 = \frac{A}{4}\,, \tag{2.8}$$

which is exactly the spherical bound.

The spherical entropy bound might lead one to wonder whether there is a generalization that holds empirically. The straightforward idea is simply to drop the assumptions of the spherical bound and postulate the conjecture:

Conjecture 2.1 (Spacelike entropy bound)**.** *In some spacetime, let V be a compact part of a hypersurface of equal time and S be the entropy of all matter systems in V. If A is the area of ∂V, then*

$$S \leq \frac{A}{4}\,. \tag{2.9}$$

as formulated in [40]. Unfortunately, it turns out that there are many counterexamples to this conjecture, which can also be found in [40]. One would be a closed universe where it is not

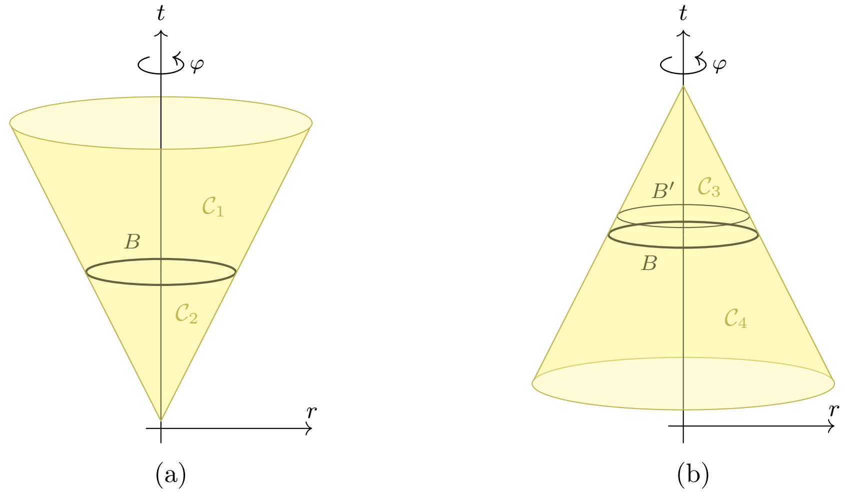

Figure 2.1: Demonstration of the notion of a light-sheet in $\mathbb{M}^3$ for the spatial surface $B = \mathbb{S}^1$. $\mathcal{C}_i$ are the four null congruences orthogonal to B. Only $\mathcal{C}_2$ and $\mathcal{C}_3$ are light-sheets, since they have negative expansion everywhere.

clear what is inside and outside a given surface. One can make the region V arbitrarily large, while keeping the boundary region arbitrarily small. Another interesting violation is based on relativistic length contraction. If one takes any 2D spacelike surface with its arbitrarily open neighbourhood in 4D spacetime, then there exists another 2D spacelike surface in the neighbourhood with arbitrarily small area [26]. This means that the spacelike entropy bound is broken even for spherical weakly gravitating systems by a non-standard choice of coordinates. Other counterexamples include gravitational collapse or an expanding universe. In the next subsection we introduce a better conjecture which evades all these problems.

2.1.3 Bousso's bound

The assumptions under which we derived the Bekenstein and spherical entropy bounds are quite strong, and we have seen that a large class of physical systems violates the conjecture of the spacelike bound 2.1. This led the physicist Raphael Bousso to introduce what is known as Bousso's bound or the covariant entropy bound. It has many advantages and is empirically very successful. In fact, it supersedes the original motivation because it holds even in situations where it is not required by the generalized second law of thermodynamics, for example in regions that cannot be transformed into a black hole (e.g. in an expanding universe).

To formulate Bousso's bound, we first need the notion of a light-sheet.

Definition 2.2. *Let us have a spacetime of dimension d with a $d-2$ dimensional spatial surface B and a null vector field k^a which is orthogonal to B and generates a geodesic congruence $\mathcal{C}$. If, in addition, the null hypersurface $L(B) := \bigcup_{\gamma \in \mathcal{C}} Im(\gamma) \setminus B$ splits into connected components, each connected to some connected component of B, and $\forall p \in L(B) : \theta(p) \leq 0$ (here θ is the expansion as defined by Definition A.6), then we call $L(B)$ a* ***light-sheet*** *of B.*

Every $d-2$ dimensional surface always has four orthogonal null directions. An example of this is illustrated in Figure 2.1. The congruences $\mathcal{C}_1$ (future directed), $\mathcal{C}_4$ (past directed) are outgoing with positive expansion and $\mathcal{C}_3$ (future directed), $\mathcal{C}_2$ (past directed) are ingoing with negative expansion. We see that in this case two of the four congruences form light-sheets. But can we always find a light-sheet for a given surface? Certainly yes, because there are always two pairs (in our example one pair is depicted in Figure 2.1a and the second in Figure 2.1b) where the two congruences of each pair are continuations of each other. So either one member of each pair forms a light-sheet or, in the case of vanishing expansion, both.

Let us now focus for example on the congruence $\mathcal{C}_3$ in Figure 2.1b. We start at B and follow each geodesic of $\mathcal{C}_3$ for an affine length $\Delta\lambda$. This gives us a new surface B', and we denote the change in 'area' (in this case the circumference) by $\Delta A = A(B') - A(B)$. It turns out that [40]

$$\frac{1}{A(B)} \frac{\Delta A}{\Delta \lambda} \xrightarrow{\Delta\lambda \to 0} \theta = \frac{1}{A} \frac{dA}{d\lambda}\,, \tag{2.10}$$

where θ is again the expansion. Nothing in this discussion really depended on our specific example, so it can easily be generalized.

It is clear that in our example the light-sheets can be understood as the 'inside' of B, but it is far from obvious that this is the case for closed surfaces in general spacetimes. The formula (2.10) motivates exactly this in the case where the expansion is negative. This is because according to (2.10), negative expansion means $\frac{dA}{d\lambda} \leq 0$, which implies that the area decreases exactly as one would expect for the 'inside'. It also verifies our intuition about what the expansion measures.

Now that we defined light-sheet, Bousso's bound is simply stated as:

Conjecture 2.3 (Bousso's bound)**.** *Let (M, g_{ab}) be a spacetime of dimension d satisfying the Einstein equations and the dominant energy condition. If additionally B is $d-2$ dimensional spatial surface, $L(B)$ any of the light-sheets of B, $S[L(B)]$ the entropy of this light-sheet and $A(B)$ the area of B, then*

$$S[L(B)] \leq \frac{A(B)}{4}\,. \tag{2.11}$$

The assumption of the dominant energy condition here is to prevent superluminal energy transport, as explained in Section A.2. In that section, one also finds that the dominant energy condition implies the null energy condition. Since k^a is orthogonal to a spacelike surface and the congruence is geodesic, we have hypersurface orthogonality of k^a on every connected component of $L(B)$. Together with the Einstein equations, this gives us the assumptions of the focusing theorem A.7 on each component, which is important later.

One can notice that this bound differs from the others in several important ways. For example, in the other bounds, we start with the bulk, which is a spatial volume, and then take its boundary. In the case of Bousso's bound, on the other hand, we start with the boundary and then find the inner region, which is not spacelike, but null. Note that in Definition 2.2, B is not required to be closed or connected as with the other bounds. Also, as for the spacelike entropy bound 2.1, but unlike for the Bekenstein bound and the spherical bound, there is no heuristic derivation of Bousso's bound, and it is at heart a conjecture, be it a very successful one. Finally, the most distinctive aspect of Bousso's bound is its use of light-sheets.

What motivates the use of light-sheets? Apart from the fact that the bound is doing very well in thought experiments [40], there is also circumstantial evidence. Some can be found in the paper [51], where Susskind suggests how in the holographic principle (which is based on entropy bounds) one maps the bulk onto a distant holographic screen using light rays. This plays well with the name 'holographic principle'. He even makes use of the focusing theorem which is essential to Bousso's bound, as is discussed below. Another light-like idea explored in this paper is the independence of the light cone quantization of string theory on the longitudinal coordinate x^- and the possible effective dimensional reduction of the theory. For constant x^-

we have a null hypersurface in Minkowski, which has everywhere vanishing expansion and thus forms a light-sheet. The use of null geodesics also plays well with AdS/CFT, since these can reach the conformal boundary, unlike the timelike geodesics.

The essence of the success of Bousso's bound may well lie in the focusing theorem. According to Definition 2.2, the part of a light-sheet connected to each connected component of B must be connected. This means that the corresponding light-sheet component is generated by starting at B and following the geodesics into the past or the future, as long as the expansion is non-positive. As discussed earlier, the focusing theorem A.7 holds, and we have two options. In the case of vanishing expansion θ and an empty universe, the light-sheet can extend indefinitely. However, this does not mean that Bousso's bound is violated, because there is no entropy, so it is trivially satisfied. If there is any matter in the path of the light-sheet component, that is $T_{ab}k^a k^b > 0$, then according to Raychaudhuri's equation A.3 θ must become negative (if it was not already). Then the second part of Theorem A.7 tells us that θ decreases until it reaches a caustic, a point where $\theta \to -\infty$. At the caustic, the light-sheet ends because curves in a congruence cannot cross and, in addition, the congruence on the other side would expand. In Figure 2.1 the caustics are the tips of the light cones, although in general caustics need not be point-like.

If we take that entropy always requires $T_{ab}k^a k^b > 0$, we see from Raychaudhuri's equation A.3 that the presence of entropy effectively accelerates the formation of caustics and therefore reduces the extent of light-sheets. It turns out that this protects Bousso's bound from being violated, but the focusing is also exactly slow enough to allow saturation of the bound in some examples [40]. The reason for this is however not fully understood.

Let us also look at how Bousso's bound relates to the other entropy bounds and the generalized second law. One of the main results is the spacelike projection theorem [40]. Suppose we have a closed spacelike surface B which allows at least one future directed light-sheet $L(B)$. If $\partial L(B) = B$ and V is a spatial region enclosed by B on the same side as $L[B]$, then

$$S[V] \leq S[L(B)] \leq \frac{A}{4}\,. \tag{2.12}$$

It is easy to see that this is just a consequence of Conjecture 2.3. This is because, regardless of our choice of V, any matter present in V will also be present in $L(B)$. The first inequality in (2.12) is a consequence of the standard second law of thermodynamics, and the second inequality is just the statement of Bousso's bound. An example where the assumptions of the spacelike projection theorem are satisfied would be in Figure 2.1. In this case, the future directed light-sheet is generated by $\mathcal{C}_3$.

The spacelike projection theorem basically gives us a region of validity of the spacelike entropy bound 2.1. With the strong conditions of the spherical entropy bound, we expect the assumptions of the spacelike projection theorem to be satisfied, so we see that the spherical entropy bound is a consequence of Bousso's bound. The Bekenstein bound is again problematic. There is a generalized version of Bousso's bound, introduced in [52], which implies the generalized second law in the Geroch process and therefore provides a stronger version of the Bekenstein bound that one would obtain from the naive classical treatment of the process, as discussed in Subsection 2.1.1. In that subsection, we also discuss that this version is clearly violated for very flat systems, which is an argument against the generalized Bousso's bound. Bousso's bound itself is not sufficient to ensure that the generalized second law is satisfied in the Geroch process [40], and unfortunately, we do not know of any derivation of the Bekenstein bound from Bousso's bound. However, Bousso's bound implies the generalized second law for any black hole formation process, as used in the heuristic derivation of the spherical entropy bound [40].

We have already mentioned the success of Bousso's bound. There is no known real-world system for which there is a danger of violating the bound, and so far it has also managed to evade (in our opinion) all thought counterexamples that we know of. That is except for the species problem discussed in Subsection 2.1.2, but we consider it of no great importance. Let us just hint at how Bousso's bound bypasses the counterexamples of the spacelike entropy bound discussed at the end of the previous subsection. We draw from [40] where one can find many other thought experiments like an expanding universe or gravitational collapse. In the case of a small closed surface in a closed universe, it is simple. Only the direction of the smaller enclosed region is the direction in which are the light rays non-expanding. The example based on the Lorentz contraction is more complicated. What ends up happening is that the light-sheet does not always cover the entire spatial bulk. This means that the assumptions of the spacelike projection theorem can be violated, and we cannot expect the spacelike entropy bound 2.1 to hold. Bousso's bound is satisfied because it turns out that as the surface area becomes smaller (the boundary becomes more light-like) the fraction covered by the light-sheet decreases exactly to counteract the effect.

Of course, Bousso's bound also has some shortcomings. For once, it is not perfectly well-defined. Defining precisely the light-sheet entropy is quite tricky. One such problem is that entropy counts the microstates of a system and for different microstates, the light-sheet might differ. In this case, the right thing to do is to fix the boundary, not the light-sheet, but in practice, it is hard to account for this. One can also generalize the entropy by including the Bekenstein-Hawking entropy. More details regarding the entropy in Bousso's bound can be found in [40]. Additionally, Bousso's bound is taken to be valid only in some semi-classical regime. This is obvious from the formulation which uses geometrical language and is therefore not fully quantum. Also, the dominant energy condition is known to be violated in general QFT. One should for example not expect the bound to hold when Hawking radiation features, because it is known to involve negative energy flux. However, Bousso's bound is not fully classical either, because when $\hbar \to 0$ it becomes trivial.

As already discussed, Bousso's bound is only an empirical law such as the second law of thermodynamics. However, large enough evidence has mounted, and it is sufficiently nontrivial observation, that we should take it very seriously. Let us also mention that a refined quantum version of Bousso's bound has been proved in some particular circumstances [53, 54]. Bousso's bound largely rest on the known laws of physics, but it goes beyond them, and it is in my opinion probable that it will reveal to us something about the nature of quantum gravity. The final theory of physics does not have to feature it explicitly, but it should be compatible with it. Indeed, it hints at the holographic principle, the development of which has been to a large extent motivated by string theory even though it is not obvious that Bousso's bound is satisfied in string theory.

2.2 Case for the holographic principle

In the previous section we discussed entropy bounds which, motivated mainly by the generalized second law of thermodynamics from Section 1.3, put a limit on the entropy content of a given region. Based on its covariant, fairly well-defined form and empirical success, we then established the superiority of Bousso's bound 2.3. In this section, we try to argue, on the basis of the entropy bound and related observations, for some features of the final quantum theory of gravity. These considerations then lead us to the so-called holographic principle. Unless noted otherwise, this discussion based on that in [40].

The Standard Model of particle physics is currently one of our best theories of the world, and it is a local QFT. Let us now present a very heuristic argument as to why a local QFT might not be the right picture. We can roughly think of a local QFT as a collection of quantum harmonic oscillators at every point in space. Every single quantum harmonic oscillator already has an infinite dimensional Hilbert space, so it comes as no surprise that the Hilbert space of QFT even in some finite volume is also infinite dimensional. We can however argue that this dimension might be effectively reduced by some simple cut-offs coming from gravity. First, we suspect there is some minimal length scale on the order of ℓ_p as discussed for example in [15], so let us discretize space into a lattice of spacing ℓ_p. We realize that this oversimplified model has many problems as it for example violates symmetries like Lorentz invariance as also pointed out in [15]. Still, some similar principles could be at work, and we are just being heuristic. Also, the energy of each oscillator at the lattice nodes should not be more than m_p because then we would create a black hole. This bounds the spectrum of each oscillator from above and provides us with a finite number of states n. Therefore, if we consider a local QFT in some region of finite volume V, we can estimate the dimension of effective Hilbert space of this QFT as n^V.

However, this simple estimate is at odds with Bousso's bound. To see why, let us first look at a conjecture that the bound suggests if we assume that reality is quantum (which is our assumption throughout this work). The state of maximum entropy on a light-sheet $L(B)$ is the maximally mixed state with density matrix $\rho = \frac{1}{\dim \mathcal{H}[L(B)]} I$ and von Neumann entropy $S = -\mathrm{Tr}(\rho \ln \rho) = \ln \dim \mathcal{H}[L(B)]$. Thus, given an entropy bound S, we can limit the Hilbert space dimension to at most e^S. This leads us to the following conjecture.

Conjecture 2.4. *In some possibly emergent spacetime, let us have $d-2$ dimensional spacelike surface B and its light-sheet $L(B)$. If $\mathcal{H}[L(B)]$ is the Hilbert space describing $L(B)$ in the final theory, then its dimension is bounded as*

$$\dim \mathcal{H}[L(B)] \leq e^{\frac{A(B)}{4}} . \tag{2.13}$$

Here we have omitted that the Hilbert space is only effective for future purposes, but for now, imagine that the word 'effective' is there. We now consider a spherical region of radius R in approximately Euclidean space (weak gravity), described by a local QFT with effective Hilbert space $\mathcal{H}_{\text{eff}}$. Furthermore, we work in a regime where the spacelike projection theorem from Subsection 2.1.3 applies to our region. According to the spacelike projection theorem, Conjecture 2.4 reduces to $\dim \mathcal{H}_{\text{eff}} \leq e^{A/4} = e^{\pi R^2}$. But for large regions, this is certainly incompatible with our previous estimate $\dim \mathcal{H}_{\text{eff}} = n^V = n^{\frac{4}{3}\pi R^3}$. This disagreement is very serious, because it even holds that

$$\frac{e^{A/4}}{n^V} = e^{-\pi R^3 \left(\frac{4}{3}\ln n - \frac{1}{R}\right)} \xrightarrow{R \to +\infty} 0 . \tag{2.14}$$

We see that when we begin by describing our universe by the Standard Model, even with the cut-offs made earlier, most of the effective Hilbert space probably has to be discarded to accommodate quantum gravity. The first cut-offs were local, but the new constraints likely point to the non-local nature of the final theory.

So far we have been conservative about the reduction of Hilbert space, using the term 'effective'. This is because one could say that some states are simply not realized due to their gravitational instability, but the final theory still features the full Hilbert space. But if we believe Conjecture 2.4 (even in the effective formulation), we have to take seriously that this is not just a practical limitation that these states would immediately collapse into a black hole, and we do not see them in nature. Even in theory, we cannot imagine exciting these states because the entropy bound cannot be violated even for a short moment. Nor can such states be excited in a black hole, since this would also violate the bound. This is because we assume that the Bekenstein-Hawking entropy really counts the microstates of a black hole, as already mentioned in Subsection 2.1.2. It seems a bit strange that most of these states are inaccessible. One might even question their physicality. Perhaps they are just a consequence of some redundant gauge degrees of freedom. Let us now take a more radical view and ask if we can find a theory that only uses the reduced Hilbert space. This is the basis of the holographic principle.

There is no single formulation of the principle, and in [40] something similar to Conjecture 2.4 is even taken as the holographic principle. We think that a more accepted version is one inspired by the papers [51, 55] which first introduced the idea. This form of the holographic principle holds that there is an equivalent description of our world by a theory with one dimension of space less, living on a 'holographic screen'. What is meant by the holographic screen may be different in different theories, but what they have in common is that there is a lower dimensional theory that uses only the reduced Hilbert space.

The name can be thought of as suggesting that the higher dimensional world is projected like a hologram from the holographic screen. This analogy is maybe most vivid in [51], where our world is mapped onto the lower dimensional screen using rays of light. This also plays well with Bousso's bound, where the light-sheet is basically a collection of worldlines of non-dispersing light rays perpendicularly coming out of the screen. Bousso even tried to use this aspect of his bound to construct holographic screens in general spacetimes [56], where the screen is in a sense obtained by tracing back the light rays.

2.3 AdS/CFT correspondence

A particularly explicit realization of the holographic principle is provided by the famous AdS/CFT correspondence, discovered by Maldacena [57] and explicitly connected to the holographic principle by Witten [58]. This conjecture posits a duality between a theory of quantum gravity, namely type IIB superstring theory (needs a 10D spacetime) in an asymptotically $\mathrm{AdS}_5 \times \mathbb{S}^5$[1] spacetime with N units of five-form flux on $\mathbb{S}^5$ (we will refer to as the AdS theory) and a quantum field theory without gravity, the $3+1$ dimensional $\mathcal{N} = 4$ (16 real supercharges) supersymmetric Yang-Mills theory with gauge group $U(N)$ (the CFT for short).

This duality can be motivated directly from string theory by considering a stack of N coincident D3-branes. The physics of this system can be described in two equivalent ways. The first is in terms of flat space open strings ending on the N D3-branes, which at low energies yield the CFT, together with a sector of flat-space closed strings. Alternatively, one can describe the system as closed strings propagating in the background of the branes, which is a spacetime containing a charged extremal black hole. In the near-horizon limit (which corresponds to a low energy limit thanks to redshift of the curved geometry), this geometry becomes $\mathrm{AdS}_5 \times \mathbb{S}^5$, again with a sector of flat-space closed strings at the asymptotic infinity.

In both descriptions, the low-energy or near-horizon limit leads to two decoupled sectors: the universal flat-space closed-string sector, and a second sector, either the CFT (from the open-string perspective) or the AdS theory (from the closed-string perspective). Since the flat-space closed-string sectors are the same on both sides and decouple, this led Maldacena to conjecture the equivalence of the remaining theories. In the limit where the rank N of the gauge group

[1] As usual with compact extra dimensions in string theory, the $\mathbb{S}^5$ does not directly affect the universal low-energy physics we focus on here (though it is crucial for the consistency and symmetry of the full background).

becomes large and the Yang–Mills coupling is strong, the dual string theory simplifies at low energies to classical supergravity in AdS. The gravitational part of the supergravity action then reduces to the Einstein-Hilbert action with a negative cosmological constant, so the familiar equations of general relativity are recovered as the leading-order description. Thus, the AdS/CFT correspondence provides a powerful tool for studying strongly coupled quantum field theories using classical gravity. We will refer to this limit as 'the classical limit' although on the CFT side it is not at all classical.

While the AdS_5/CFT_4 correspondence is the canonical example, the AdS/CFT duality has been generalised in several important directions. In particular, it is now understood that similar dualities hold for a wide class of dimensions, giving rise to the AdS_{d+1}/CFT_d correspondence [59], where a quantum gravity theory in (d+1)-dimensional AdS space is dual to a d-dimensional conformal field theory. Important examples are AdS_4/CFT_3 [60] and AdS_3/CFT_2. The AdS_3/CFT_2 case is special since quantum gravity in 2+1D is exactly solvable [61] and 1+1D CFTs are particularly well-understood and classified [62]. In fact, even before the advent of the AdS/CFT correspondence, Brown and Henneaux [63] showed that the asymptotic symmetry group of AdS_3 gravity with suitable boundary conditions is the infinite-dimensional conformal group in two dimensions, just like the symmetry group of a 2D conformal field theory. Very interesting is also the lowest dimensional case where we have what has been called by Maldacena [64] the $NAdS_2/NCFT_1$ duality. N is denoting here 'nearly' as the gravity on the AdS side is the dilaton Jakiw-Teitelboim gravity [65, 66] and on the CFT side, we do not have a true CFT but a quantum mechanical system with emergent conformal symmetry. This so-called Sachdev-Ye-Kitaev (SYK) model [64, 67, 68] plays an eminent role in the so-called AdS/CMT correspondence discussed later. AdS_2 can also be used to describe the near horizon geometry of AdS extremal black holes [69, 70].

Moreover, although the original duality was discovered in the context of highly supersymmetric theories, many aspects of the correspondence are believed to extend to less or even non-supersymmetric settings, and explicit examples with reduced or broken supersymmetry are known [59]. There have also been attempts to generalise the holographic correspondence to other backgrounds, such as de Sitter space [71], but these encountered difficulties, and the role of AdS geometry appears to be crucial for the correspondence. In what follows, we illustrate everything on the AdS_5/CFT_4 correspondence, but most of the conceptual structure extends to these broader classes of dualities [59].

The correspondence is made precise by the so-called GKPW (Gubser, Klebanov, Polyakov and Witten) prescription [58, 72], which provides a dictionary between bulk and boundary observables by relating the generating functional of the CFT to the partition function of the AdS theory with prescribed boundary conditions. In general, this is expected to hold for the full theories (full string theory of the AdS side), but explicit computations are typically possible only in certain limits. In the classical limit the partition function on the AdS side is dominated by its classical saddle, and the GKPW prescription takes the following approximate form for a scalar field:

$$\left\langle \exp\left(\int d^4x J(x)\mathcal{O}(x)\right)\right\rangle_{\mathrm{CFT}} = \int_D \mathcal{D}\phi\, e^{iN^2 S_{\mathrm{AdS}}(\phi(x,r))}$$

Here, $\mathcal{O}(x)$ is a local, gauge-invariant operator in the CFT, and $J(x)$ is an external source coupled to this operator. D is the domain of integration $D = \left\{\phi \mid \phi(x,r) \xrightarrow{r\to\infty} J(x)\right\}$ (r is the AdS radial coordinate defined in the next paragraph) which indicates that we integrate only over field configurations with a given boundary condition. The left-hand side is the generating functional of the CFT, while the right-hand side is the path integral of a bulk scalar field on the AdS background, usually evaluated in the semiclassical approximation due to the large N^2 factor in the exponent, which justifies the use of the stationary phase approximation,[2] so that the path integral is dominated by the classical solution. For example, a scalar field of mass m in AdS is dual to a local, gauge-invariant scalar operator of scaling dimension Δ, related by $m^2L^2 = \Delta(\Delta - 4)$ (L being the AdS radius also defined later). More generally, the GKPW prescription applies to bulk fields transforming under arbitrary representations of the Lorentz group. For instance, the bulk metric $g_{\mu\nu}$ corresponds to the boundary energy-momentum tensor operator $T_{\mu\nu}$.

There are basically two further regimes for the duality. The little bit simpler regime is when we look just into the Poincaré patch of the global AdS spacetime. There the AdS metric can be written in the Poincaré coordinates as

$$g_{\mathrm{AdS}_5} = \frac{L^2}{z^2}(g_{\mathbb{M}^4} - dz^2)\,, \tag{2.15}$$

where $L \in (0, \infty)$ is the so-called AdS radius, $g_{\mathbb{M}^4}$ is the 3+1D Minkowski metric and $z \in (0, \infty)$ is referred to as the radial coordinate. This form of the metric is nice for example in that one sees that in the limit $z \to 0$ of the conformal boundary, the spacetime is conformally equivalent to the Minkowski spacetime. In fact each surface of constant z is just $\mathbb{M}^4$ with a different scale factor and the Poincaré patch can be foliated by these. Using quite complicated coordinate transformations, one can then transfer to global coordinates in which the metric reads

$$g_{\mathrm{AdS}_5} = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2 d\Omega_3^2 \quad \text{with} \quad f(r) = 1 + \frac{r^2}{L^2}\,, \tag{2.16}$$

$d\Omega_3^2$ being the metric of $\mathbb{S}^3$ and $r \in [0, \infty)$ being here the radial coordinate. With this form of the metric, one instead sees that the conformal boundary is conformally equivalent to $\mathbb{S}^3 \times \mathbb{R}$ and the spacetime can be again foliated by surfaces of constant r, which correspond to spheres of different radii.

It then turns out that the duality holds not only between the string theory in global AdS and the CFT on $\mathbb{S}^3 \times \mathbb{R}$ but also when we restrict ourselves just to the Poincaré patch and take the CFT as living in the Minkowski spacetime. This is one of the reasons why it makes sense to think of the CFT as really physically living on the conformal boundary of the AdS, and we thus refer to these as the 'bulk' (AdS) and 'boundary' CFT. This is of course not the only reason, adopting this interpretation can give many more intuitions. For example, one can interpret the asymptotic form of the bulk fields in the GKPW prescription as really being the sources at the boundary.

A key feature of the correspondence is the relation between the extra dimension of AdS and the energy scale in the CFT. The radial coordinate can be understood as emergent, encoding the energy scale of the boundary theory. Points near the boundary correspond to the UV regime, while points deeper in the bulk (large z or small r) correspond to the IR regime on the CFT side. In fact, the whole renormalization group (RG) flow in the CFT can be viewed geometrically as movement along the radial direction in AdS. Introducing a cut-off in the bulk by moving the holographic screen away from the AdS boundary, for example, in the Poincaré patch to $z = \epsilon$

[2]The stationary phase approximation states that for integrals of the form $\int_{\mathbb{R}^n} e^{iN^2 S(\phi)} d^n\phi$ with a single non-degenerate critical point ϕ_{cl}, the leading asymptotic behaviour as $N \to \infty$ is $\mathcal{N} e^{iN^2 S(\phi_{cl})} + o(N^{-n})$, where $\mathcal{N}$ is a constant depending on the Hessian of $S(\phi)$ at ϕ_{cl}. In this context we use notation reminiscent of the problem at hand, where one deals with a functional integral, which formally generalises this idea for $n \to \infty$. The saddle-point ϕ_{cl}, where the action $S(\phi)$ is stationary then corresponds to a classical solution of the equations of motion.

corresponds to introducing an ultraviolet cut-off $\Lambda = \frac{1}{\epsilon}$ in the CFT. This is quite an explicit realization of coarse-graining foliating holographic screens that we encounter in Section 3.2.

The AdS/CFT correspondence realizes many of the emergent gravity ideas discussed in previous sections in a very explicit and calculable way. For example, one can heuristically count degrees of freedom on both sides introducing some cut-off (corresponds to considering a region only inside the corresponding holographic screen on the AdS side) and see that they match, as discussed at the end of [40]. This is in the spirit of the spherical entropy bound of Section 2.1.2 when we have a cut-off in the global AdS bulk but in the limit only light-like geodesics can reach the conformal boundary so it seems more like a manifestation of the full Bousso's bound of Subsection 2.1.3 of which we have seen that the spherical entropy bound is a special case (see (2.12)). It also fits with the idea of projecting bulk from the boundary using light rays, the dispersal of which can be maybe very heuristically thought of as corresponding to the coarse-graining on the holographic screens, which we discussed happens as we delve deeper into the bulk.

In the regime of the classical limit one can look at which states of the CFT correspond to black holes in the AdS. In the Poincaré patch it turns out that the corresponding solutions of Einstein equations are actually black holes with a planar horizon ($\mathrm{AdS}_2 \times \mathbb{R}^3$ near horizon geometry) and these correspond to thermal states in the CFT. One could maybe even guess that based on that we know black holes to follow laws very reminiscent of thermodynamics (see Section 1.1). Now we see that not only are the laws of black hole dynamics reminiscent of thermodynamics, but in the AdS/CFT correspondence there is actually a duality mapping the black hole dynamics to actual thermodynamics of the boundary CFT [59, 73]. In this regime, one can also study charged extremal planar black holes, which, as discussed in Section 1.1, violate Nernst's formulation of the third law of thermodynamics. One can then ask whether a corresponding state violating this law exists on the CFT side. Indeed, such a state has been identified in [74][3] however, these configurations have also been found to be extremely unstable. Moreover, if one instead takes the full global AdS one now has a scale coming from the AdS radius L which allows a phenomenon known as the Hawking-Page phase transition [75], in which there is a transition from a thermal gas in AdS at low temperatures to an AdS black hole at high temperatures. On the boundary field theory side, this is interpreted as a confinement-deconfinement transition, which is a well-known feature in gauge theories. At low temperatures, the CFT is in a confined phase, while at higher temperatures it transitions to a deconfined phase described by a black hole in the bulk.

Although the AdS/CFT correspondence arose as a highly theoretical construction, it has found concrete applications in several areas of physics, most notably in the study of strongly coupled systems in condensed matter theory, in modelling QCD-like gauge theories, and in quantum information. In recent years, these applications have become so extensive that entire textbooks are devoted to the subject [17, 18].

In condensed matter physics, the correspondence is often called AdS/CMT. In this context, the duality is used in the direction opposite to quantum gravity research. By taking the classical limit one can study the strongly coupled, high energy regime of certain quantum field theories that are impossible to analyse using standard perturbative techniques. This is especially relevant in condensed matter systems where the usual quasiparticle paradigm breaks down and new non-perturbative theoretical tools are needed, which are central to modern condensed matter research. The AdS/CFT duality provides a framework in which transport properties, phase transitions, and other phenomena of strongly interacting systems can be studied using the tools of classical

[3]The bulk gauge field A_μ corresponds, via the GKPW prescription, to a conserved current operator J_μ in the boundary CFT. The dual field theory state can thus be interpreted as a thermal state with constant charge density and background magnetic field.

general relativity. The duality is reminiscent of ideas such as the Kramers–Wannier duality or the more general concept of weak-strong coupling dualities, where an intractable regime of one theory is mapped to a more tractable regime of another. Applications of AdS/CMT include the study of strange metals and systems that exhibit non-Fermi liquid behaviour. One particularly influential example is the SYK model, which as already mentioned realizes $NAdS_2/NCFT_1$ duality. It is a solvable model of N randomly interacting Majorana fermions and is believed to capture some universal features of strongly coupled quantum systems [64, 67, 68]. This model has become a valuable testing ground for many ideas such as quantum chaos [76].

Another important development related to AdS/CMT is the study of hydrodynamics and its connection to the dynamics of black hole horizons. The so-called membrane paradigm provides an effective description of black hole horizons as if they were dynamical membranes with physical properties such as viscosity and conductivity [12]. In a manner similar to black hole thermodynamics, this idea was developed even before the AdS/CFT correspondence. Then it was a tool that allowed physicists to describe horizon dynamics in terms familiar from fluid dynamics. Later, it was realized that the membrane paradigm naturally connects to the study of strongly coupled quantum field theories through holography. The AdS/CFT correspondence maps the dynamics of horizons in gravity to the hydrodynamics of the dual field theory, a subject now often called holographic hydrodynamics [77], thus providing a realization of the analogy of the membrane paradigm (similarly as black hole thermodynamics found realization in AdS/CFT). This approach led to the famous result for the ratio of shear viscosity to entropy density in strongly coupled plasmas, which was found to saturate a universal bound [78]. Further developments [79] established that the dynamics of horizons in gravity can be interpreted as a real instance of hydrodynamics, thus providing additional support for the emergent nature of gravity. Holographic hydrodynamics is also covered in [17].

Because there is no natural small parameter in QCD in which to do perturbation theory, 't Hooft had the ingenious idea to expand not in the coupling, but in the rank of the gauge group, N [80]. In this so-called large N expansion, the Feynman diagrams of the gauge theory can be organized according to their topology, where each diagram can be viewed as a partition of a two-dimensional surface. It turns out that this expansion looks very much like the expansion one gets in string theory, where the diagrams of string interactions are also two-dimensional surfaces of different genus. This suggested that QCD, or at least some similar gauge theories, could be described by a string theory. However, it was quickly realized that any consistent string theory inevitably predicts a massless spin 2 particle, the graviton, which is absent in QCD. The situation changed with the AdS/CFT correspondence, where one manages to avoid the existence of spin 2 particles by having the dual string theory live in one higher dimension than the gauge theory, and the graviton in the bulk does not correspond to a physical particle in the boundary theory. In the approach called AdS/QCD, people construct models in higher-dimensional anti-de Sitter space that try to capture some important features of QCD, such as confinement, while still keeping the field theory on the boundary free of gravity [81, 82]. In this way, the ideas from string theory have returned to the study of strong interactions, but now in a form that avoids the problems that made the original string theory approach to QCD unworkable. More can also be found in [18].

One of the most striking results in the development of the AdS/CFT correspondence is the discovery of a geometric formula for the entanglement entropy in the boundary CFT, known as the Ryu-Takayanagi formula [83]. Calculating entanglement entropy is notoriously difficult in quantum field theory, and even in highly symmetric cases, such as $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, exact results are limited to one spatial dimension. The formula posits that in the classical limit the entanglement entropy of a region in the boundary theory can be computed as the area of a minimal hypersurface in the bulk geometry. More precisely, the boundary of this minimal surface must coincide with the boundary of the chosen region in the CFT, and the minimal surface is required to be spacelike and of codimension two. For example, in the standard AdS_5/CFT_4 correspondence, the entangling region is three-dimensional and the minimal hypersurface is also three-dimensional in the bulk. Due to the warped geometry of AdS space, the area of the bulk minimal surface is dominated by the region near the boundary, where it closely tracks the boundary of the entangling region. As a result, the leading divergence of the minimal area reproduces the standard area law for entanglement entropy in quantum field theory, rather than a volume law. It is also notable that the formula has the same structure as the Bekenstein-Hawking entropy formula (1.36), which will play an important role in the next chapter, particularly in Section 3.1. It also fits naturally with the holographic bounds of Section 2.1, though it essentially corresponds to their saturation. In the paper [84], the authors also consider a covariant version of the Ryu-Takayanagi formula which is very much in the spirit of Bousso's bound, using light-sheets. The similarity to the Bekenstein-Hawking formula is not that surprising, once one looks at the special case where the entangling region is taken to be the whole boundary in a thermal state. In that case, the minimal surface reduces to the black hole horizon, and the formula then gives exactly the black hole entropy. Quantum corrections to the Ryu-Takayanagi formula have also been studied [85], where it was shown that the leading correction comes from the bulk entanglement entropy across the minimal surface. The study of holographic entanglement entropy also links AdS/CFT with the theory of quantum information, in which the correspondence has found further applications that we do not discuss here.

In summary, the AdS/CFT correspondence is a central example of the holographic principle at work, and it serves as an explicit model where spacetime and gravity are seen to emerge from a lower-dimensional, non-gravitational quantum theory. The details of these topics are unfortunately beyond the scope of this section, but I hope to elaborate on them in further work and the reader can find extended discussion on them in the literature to which I have referred throughout.

Chapter 3

Theories of emergent gravity

3.1 Jacobson's emergent gravity

One of the most famous approaches to letting Einstein's gravity emerge is that of Jacobson [19]. In this approach, he derives Einstein equations as a thermodynamical equation of state. The derivation is based on associating to each local Rindler horizon an entropy proportional to its area and a temperature identified with the Unruh temperature seen by uniformly accelerated observers just inside the horizon. Then one just makes use of the fundamental relation of thermodynamics

$$T\delta S = \delta Q\,. \tag{3.1}$$

The left side of this equation clearly relates to the geometry since it is proportional to an area variation and thus essentially becomes the geometric left side of Einstein equations. The right side is the heat flow, which can be in turn expressed in terms of the energy-momentum tensor, giving rise to the right side of the equations. Of course, all of this assumes equilibrium thermodynamics, and we will shortly discuss this assumption. So one could also say that in this approach, the Einstein equation emerges from thermodynamics of local Rindler horizons. Apart from the original Jacobson's paper [19] I benefited in this subsection from the pedagogical treatment of [86] where one can find extended discussion on the assumptions which we will also touch on.

Let us set up the stage. As our considerations are local we pick an arbitrary point p of our spacetime and invoke the equivalence principle to locally treat it as flat. Then in this neighbourhood we consider a piece of a codimension 2 spacelike surface $\mathcal{P}$ such that $p \in \mathcal{P}$. We also introduce a past directed light-sheet (Definition 2.2) of $\mathcal{P}$ denoted $L(\mathcal{P})$ and choose $\mathcal{P}$ such that $L(\mathcal{P})$ will have vanishing expansion θ and shear $\hat{\sigma}_{ab}$ (Definition A.6) to first order in the distance from p. On the light-sheet we choose an affine parameter λ such that p is at $\lambda = 0$ and increases towards the future. We define also a tangent field to the congruence $k^a = \frac{d}{d\lambda}$. In our local patch of flat spacetime we introduce some spatial foliation $\Sigma(t)$ with a time coordinate $t \in [-\delta, 0]$ such that $\lambda = 0$ corresponds to $t = 0$ and δ is sufficiently small for all our considerations. Then we define $\mathcal{P}' := L(\mathcal{P}) \cap \Sigma(-\delta)$ and $\mathcal{H} := L(\mathcal{P}) \cap \Sigma([-\delta, 0])$ so that we are now able to get the area variation as:

$$\delta A := A(\mathcal{P}) - A(\mathcal{P}') = \int_{\mathcal{H}} \theta \, d\lambda dA\,, \tag{3.2}$$

where dA is the area element so that $A(\mathcal{P}) = \int_{\mathcal{P}} dA$.

Now since we have locally a Minkowski spacetime we have the symmetry to consider $\mathcal{H}$ to be a past Killing horizon with respect to a boost Killing vector field χ^a. This is the Rindler spacetime as considered in Section 1.2 where the boost Killing vector field was given in (1.20). At the horizon χ^a becomes null and can be written as $\chi^a = -\kappa\lambda k^a$ with κ being the surface gravity of the horizon as defined in Definition A.9.

Assuming local equilibrium, we now look at the thermodynamic quantities. First let us look at if it is reasonable to associate a temperature to the Rindler horizon. From Chapter 1 it should be clear that one can associate temperature to black hole horizons and we have also seen in Section 1.2 that for static observers hovering right above the event horizon of a black hole, the event horizon coincides with the Rindler horizon. We also know that for the vacuum state of a general interacting quantum field theory, the Bisognano-Wichmann theorem guarantees that the density matrix restricted to the wedge $z > |t|$ of Minkowski spacetime is that of a thermal state with respect to the boost Hamiltonian [39, 87] so that observers with constant acceleration along the z-axis will observe thermal radiation with the Unruh temperature (1.32) (now $\alpha = \kappa$ as κ would give exactly the four-acceleration of Rindler observers). These observers also observe a horizon at the boundary of the wedge, existence of which might be thought of as essential for the existence of the thermal radiation so one might as well associate to it the Unruh temperature. The heat flow through the horizon as viewed by the Rindler coordinates is then given simply as

$$\delta Q = \int_{\mathcal{H}} T_{ab}\chi^a \, d\Sigma^a = -\kappa \int_{\mathcal{H}} \lambda T_{ab} k^a k^b \, d\lambda dA \,. \tag{3.3}$$

Now maybe the biggest leap of faith is requiring that the entropy variation can be obtained as

$$\delta S = \eta \delta A \,, \tag{3.4}$$

for some universal constant η. We will see that η can be taken as defining our Newton's constant G. However, to be consistent with observations η should be very close to $1/4$ (in Planck units) so we must basically argue for the form of Bekenstein-Hawking entropy (1.36). We have again seen in Chapter 1 that this is a very natural assumption for a black hole horizon which, as already mentioned, coincides with the Rindler horizon for close by static observers. As black holes are observer-independent objects (see Definition A.8) one might fear that it would not be possible to associate entropy to observer-dependent horizons like the Rindler horizons. However, a source of hope for us may be, for example, that the formula (1.36) is believed to apply in general to apparent horizons of the FLRW (Friedmann, Lemaître, Robertson and Walker) spacetime [88–91], where for the case of the cosmological horizon of de Sitter space it was suggested already by Gibbons and Hawking [92] and these are all observer-dependent.

As with the Bekenstein-Hawking entropy, it is very hard to see what exactly it would be an entropy of and its microscopic origin. This is one of the main topics of the paper [86]. Since $\mathcal{H}$ can be thought of as a section of a light-sheet, one can think of the entropy variation as being something like $\delta S = S[L(\mathcal{P})] - S[L(\mathcal{P}')]$ and requiring (3.4) is something like requiring local saturation of Bousso's bound (Conjecture 2.3). There is also a stronger version of Bousso's bound [40, 52] where we truncate the light-sheet at some area A' and postulate a bound

$$S[\mathcal{H}] \leq \frac{\Delta A}{4} \,, \tag{3.5}$$

where $\Delta A = A - A'$ and $S[\mathcal{H}]$ is entropy passing through the truncated light-sheet $\mathcal{H}$. We used this notation since in our case, the truncated light-sheet would be exactly $\mathcal{H}$. So if we instead took $\delta S = S[\mathcal{H}]$, we would need a saturation of this version of the bound.

Classically, it is in general basically impossible to saturate Bousso's bound [93] using the known physics. However, entanglement entropies can be very large, so one could expect that gravity could emerge from quantum entanglement structure. This is also motivated from the AdS/CFT correspondence where we have discussed in Section 2.3 that the Ryu-Takayanagi formula can relate entanglement entropy of a region in the CFT to an area of extremal surface in the bulk with exactly the form of Bekenstein-Hawking entropy formula (1.36). It has been also argued that entanglement entropy can be a source of entropy for black holes [94, 95]. In [19] Jacobson points out that a large contribution could come from vacuum entanglement entropy, which is in a local QFT proportional to the area of the region considered. This has strong support in the need for the prefactor η being proportional to $1/G$ which has exactly the renormalization behaviour of vacuum entropy as discussed near the end of Section 1.3. However, there are problems with having here as the entropy entanglement entropy of a QFT which has been pointed out in the paper [86]. Even if the entropy would be proportional to the area they found a bound on the value of the prefactor η which is inconsistent with the value we need to recover general relativity and the value of the prefactor is also expected to be theory specific, not universal. One would expect some quantum gravity degrees of freedom beyond the standard QFT considerations that could provide the missing entropy, but the authors argue based on the results of [96], that in quite general circumstances relying for example on strong subadditivity of von Neumann entropy it would not be enough. That is all of this has been for the entanglement entropy of $\mathcal{H}$ but if we assume subadditivity we have $S[L(\mathcal{P})] - S[L(\mathcal{P}')] \leq S[\mathcal{H}]$. However, we can still imagine the effects of quantum gravity to be for example highly non-local (expected for example based on the discussion in Section 2.2) violating these assumptions and maybe also generalized entropies violating the strong subadditivity like the quantum Tsallis entropy would be more appropriate in such circumstances.

The last ingredient we will need comes from the Raychaudhuri's equation (A.3). Since we are dealing with a light-sheet we have hypersurface orthogonality and the twist $\hat{\omega}_{ab}$ vanishes [23]. We have chosen the $\mathcal{P}$ so that θ and σ_{ab} vanish there and since all of $\mathcal{H}$ is in a small neighbourhood of $\mathcal{P}$ we can neglect θ^2 and $\hat{\sigma}_{ab}\hat{\sigma}^{ab}$ as higher order contributions compared with the last term. Integrating (A.3) we then obtain $\theta = -\lambda R_{ab}k^a k^b$. So putting this into (3.2), that in turn into (3.4) and using (1.32) we have for the right side of (3.1)

$$T\delta S = \frac{\kappa\eta}{2\pi}\delta A = -\frac{\kappa\eta}{2\pi}\int_{\mathcal{H}} \lambda R_{ab}k^a k^b \, d\lambda dA \,. \tag{3.6}$$

Using 3.1 and (3.3) we can then write

$$\int_{\mathcal{H}} \lambda R_{ab}k^a k^b \, d\lambda dA = \frac{2\pi}{\eta}\int_{\mathcal{H}} \lambda T_{ab}k^a k^b \, d\lambda dA \,. \tag{3.7}$$

Since the direction of $\mathcal{P}$ was chosen arbitrarily, this holds for all null k^a and gives us

$$R_{ab} + f g_{ab} = \frac{2\pi}{\eta}T_{ab} \,. \tag{3.8}$$

The energy-momentum conservation $\nabla^a T_{ab} = 0$ implies $\nabla^a R_{ab} = -(\nabla^a f)g_{ab} = -\nabla_b f$ and, using the contracted Bianchi identities $\nabla^a R_{ab} = \frac{1}{2}\nabla_b R$, we finally get the Einstein equations in their full glory

$$R_{ab} - \frac{1}{2}R g_{ab} + \Lambda g_{ab} = \frac{2\pi}{\eta}T_{ab} = 8\pi T_{ab} \,, \tag{3.9}$$

where in the last equality we see that to recover Einstein equations as we know them we must have $\eta = 1/4$ which can be thought of as a definition of the Newton's constant G. Specifically we might in SI units define $G := \frac{k_B c^3}{4\hbar\eta}$.

We know that solutions to the Einstein equations include gravitational waves. In Jacobson's emergent view of gravity, gravitational waves are analogous to sound waves in that they are collective excitations, or 'hydrodynamic' modes, arising from an underlying microscopic structure. They represent macroscopic, low-energy disturbances that propagate at a fixed speed, governed by an effective equation of state. Just as a sound field is only a statistically defined observable within the fundamental phase space of a multiparticle system, and is therefore not appropriately subject to canonical quantization as if it were a fundamental field, despite the underlying molecules being quantum, the analogy suggests that it may also be inappropriate to canonically quantize the Einstein equations, even if they ultimately describe quantum phenomena.

Of course, all of our considerations assumed thermodynamic equilibrium and, to some extent, the presence of a thermodynamic limit. Both of these assumptions can fail in extreme situations. By the thermodynamic limit, I mean that even when working in a local neighbourhood, this region must still be large enough for the coarse-grained description of thermodynamics to be a good approximation. We expect the microscopic scale of the system to be on the order of ℓ_p, so in areas where the curvature radius is not much smaller than ℓ_p, these arguments would likely cease to hold. This matches our expectations, as we already know that general relativity is expected to break down in regions of extreme curvature, such as near the Big Bang or the singularity of a black hole. The local equilibrium assumption can also fail. For instance, when the frequency or intensity of a sound wave becomes high enough, local equilibrium no longer holds, entropy increases, and sound waves do not propagate in a time-reversal-invariant way. In the same way, it is reasonable to expect that if the frequency or amplitude of disturbances in the gravitational field is high enough, the Einstein equation would no longer apply, not because the quantum nature of the metric becomes important, but because the local equilibrium condition is violated. Some discussion of non-equilibrium considerations within this framework can be found, for example, in [97].

The crucial role of horizons in general relativity has been also explored in many other works, see for example [12, 98]. In [98] one can find further justification for some of the principles used in Jacobson's approach (like assigning temperature or entropy to horizons) but also, for example, how one can obtain the Einstein-Hilbert action from the thermodynamics of horizons. Another interesting work of similar spirit is [89–91] where one can find how one can obtain Friedmann equations from the first law of thermodynamics applied to apparent horizons. This is a special case of an approach described in Section 3.3. I recommend also the famous review [99] which further discusses thermodynamic aspects of gravity before the work of Verlinde which is the subject of the next section.

Another quite different approach that Jacobson came up with is based on entanglement entropy [100]. This is a very interesting approach which is definitely worthwhile to mention. The paper [101] even argues that it may be the best defined approach to emergent gravity. Let us thus give a rough overview of its techniques. It was inspired by the papers [102, 103] where they use a similar approach to derive in AdS/CFT the bulk Einstein equations linearized around the AdS background. In this Jacobson's approach one considers for each point of the spacetime at hand a spatial hypersurface Σ in which one looks at a ball $B \subset \Sigma$ centred at p. If $|\psi\rangle$ is the quantum state of Σ one defines $\rho_B = \mathrm{Tr}_{\Sigma \setminus B} |\psi\rangle \langle\psi|$ to get the entanglement entropy $S_B = -\mathrm{Tr}(\rho_B \ln \rho_B)$. One also assumes that the IR and UV regimes are decoupled. Based on that one can define $\rho_{IR} = \mathrm{Tr}_{UV}\, \rho_B$ by tracing the UV degrees of freedom. Similarly one gets S_{IR}, ρ_{UV} and S_{UV}. One then does variations of the spacetime geometry and ρ_B with the condition that the volume of B is fixed. The approach then rests roughly on these assumptions:

- $\delta S_B = \delta S_{IR} + \delta S_{UV}$ (there is minimal entanglement among degrees of freedom at widely separated energy scales)
- $\delta S_B = 0$ (the entanglement entropy is stationary with respect to these variations)
- $\delta S_{UV} = \eta\, \delta A$, where η is some universal constant and A is the area of ∂B (local holography)
- in 4D spacetime $\delta \langle K \rangle = 2\pi \mathcal{N}\, \delta(\langle T_{00} \rangle + \langle X \rangle\, g_{00})$, where $\mathcal{N}$ is just some constant, K is the modular Hamiltonian defined through $\rho_{IR} = \frac{e^{-K}}{\operatorname{Tr} e^{-K}}$, T_{ab} is the energy-momentum tensor operator, g_{ab} is the metric and X is some scalar field operator ($\delta \langle K \rangle$ is for a general QFT given by this expression)
- for sufficiently small B the geometry of the domain of dependence of B (see Definition A.3) is that of a maximally symmetric spacetime (that is Minkowski, de Sitter, or anti-de Sitter)

It can then be shown that $\delta S_{IR} = \delta \langle K \rangle$ and thus one has

$$\eta \mathcal{N} \left(G_{00} + f g_{00} \right) = -\eta\, \delta A = -\delta S_{UV} = \delta S_{IR} = \delta \langle K \rangle = 2\pi \mathcal{N}\, \delta(\langle T_{00} \rangle + \langle X \rangle\, g_{00})\,, \tag{3.10}$$

where G_{ab} is the Einstein tensor, f is some constant, and the first equality is a geometrical fact making use of the spacetime being locally maximally symmetric (last assumption). One requires this to hold for all possible spatial slicings from which one gets the equation (3.10) for all the components. Imposing energy conservation $\nabla^a T_{ab} = 0$ then fixes $f = \frac{2\pi}{\eta} \delta \langle X \rangle + \Lambda$ for some constant Λ. And finally, one has the full Einstein equations, Λ defining the cosmological constant and η defining Newton's constant G. The approach of the papers [102, 103] is similar in spirit. Using the AdS/CFT correspondence, they take B in the boundary CFT, but the area is taken instead as the area of the minimal surface in the bulk from the Ryu-Takayanagi formula described in Section 2.3. They thus derive Einstein equations in the bulk instead of the boundary where B is located.

3.2 Entropic gravity

Another approach to the emergence of the Einstein equations is that of Verlinde [20]. This approach considers gravity also as an emergent force but instead of the equation of state, it views it as an entropic force. By this, he means that it is a force coming entirely from the statistical tendency of the system to tend toward a higher entropy state and maximize entropy at equilibrium. While the usual forces come from a gradient of potential energy, entropic forces can arise without any energy gradient and are instead driven by a gradient of the entropy. So entropy, which quantifies the accessible information about the systems, acts here as a sort of potential. Familiar examples of such forces are, for instance, the force that forces polymers into coiled configurations or the force that drives osmosis through a semipermeable membrane. In the case of a polymer, one can imagine having the polymer immersed in some solution of temperature T and wanting to keep it stretched. Since there are many more compact configurations of roughly equal energy for the polymer than stretched ones, the solution will cause an effective pull on the polymer, which can be shown to, in this case, obey Hook's law [20]

$$F \propto T\, x\,, \tag{3.11}$$

where x is the average displacement from the curled equilibrium configuration.

Let us first remind the reader of some thermodynamics. If we want to describe a system that exerts on its surroundings average entropic force F over the infinitesimal displacement of the generalised coordinate dx, the first law of thermodynamics reads

$$dU = TdS - F \cdot dx\,. \tag{3.12}$$

The term $F \cdot dx$ denotes the work done by the system, and the dot indicates that we project the force onto the direction of the displacement. The generalized coordinate x may take many forms, and in the example of the polymer, we have seen it as the average displacement from equilibrium. This also shows that F acts as the generalized force conjugate to x, and the first of Maxwell relations,

$$\left(\frac{\partial S}{\partial x}\right)_U = \frac{F}{T} \qquad \left(\frac{\partial S}{\partial U}\right)_x = \frac{1}{T} \tag{3.13}$$

makes it clear that the entropic force is at constant energy proportional to the rate at which entropy increases with x and gives us the entropic force as

$$F = T\left(\frac{\partial S}{\partial x}\right)_U\,. \tag{3.14}$$

The entropic force thus points in the direction of increasing entropy and is proportional to the temperature. The subscript again stresses that the derivative is taken at fixed energy. Thus we see what we hinted at at the beginning; at the macroscopic level and for fixed temperature, such forces can be described as conservative, with an effective potential emerging from the entropy landscape. However, this potential does not correspond to any microscopic energy; rather, it is emergent, rooted in the statistical tendency of the system to maximize entropy.

This can be studied in more detail in the micro-canonical ensemble, because it takes the total energy into account including that of the heat bath. There we have the famous Boltzmann's formula for the entropy of a system at fixed total energy U of the combined system (system + bath)

$$S(U, x) = \ln \Omega(U, x) \tag{3.15}$$

where $\Omega(U, x)$ denotes the volume of the configuration space for the entire system as a function of U and the generalised coordinate x. From a statistical mechanics viewpoint, the dependence of entropy on x is entirely configurational; there is no microscopic energy contribution associated with x, only a change in the number of accessible states. To determine the entropic force, one again introduces a generalized force F exerted by the system on the bath and examines the balance of forces. Specifically, one considers the micro-canonical ensemble given by $\Omega(U - Fx, x)$, and imposes that the entropy is extremal

$$0 = \frac{d}{dx} S(U - Fx, x) = -\frac{\partial S}{\partial U} F + \frac{\partial S}{\partial x}\,. \tag{3.16}$$

If one uses the thermodynamical definition of temperature $T = \left(\frac{\partial U}{\partial S}\right)_x$ one gets back the (3.13) but with the caveat that temperature and force may microcanonically become dependent on position and energy. The term Fx can be viewed as the energy expended by the system for getting to the state with generalized coordinate x. This equation tells us, therefore, that the energy of the heat bath is decreased when the system returns to a state with generalized coordinate $x = 0$ (e.g. polymer slowly returns to its equilibrium position), but that the entropy stays the same. In this sense, the force acts adiabatically.

The best familiar system for analogy to Verlinde's approach to emergent gravity is probably the already mentioned case of osmosis through a semipermeable membrane. Membrane description has also shown to be very useful for describing the event horizons of black holes [12] and, of course, the black hole horizon is also just semipermeable. In this approach, we will use as membranes what we will call holographic screens. These will be codimension 2 spatial surfaces where we will imagine on the inside of the screen having the microscopic description without gravity and on the outside already emergent spacetime geometry. To this screen, we then associate entropy of the microscopic degrees of freedom on the side that is yet to emerge. By the virtue of the holographic principle (see Section 2.2) we can maybe think of the information as really being encoded on the holographic screen.

Now a crucial step in Verlinde's approach is basically requiring that the Bekenstein bound (see Subsection 2.1.1) is saturated for a single fundamental particle. So we will assume equality in the formula (2.1) where E is the total energy of the system at the asymptotic infinity where we assume the spacetime to be flat. For a single particle we thus have $E = mc^2$ where m is the rest mass of the particle. In Bekenstein bound R is taken to be the radius of the smallest circumscribing sphere of the system also at rest at the asymptotic infinity. In [48] Bekenstein derived this bound from imagining lowering the system into a black hole and then requiring the generalized second law of thermodynamics (see Section 1.3) to hold. Now we instead imagine lowering the system towards the holographic screen. In the derivation of the generalized second law we considered the system to be part of the black hole when it was about proper distance R from the horizon, now we consider the particle to cross the holographic screen and merge with the microscopic degrees of freedom if it comes closer to it than R. When the particle enters past the holographic screen there is a corresponding change in entropy of the screen equal to the entropy of the particle and if we change notation and write $\Delta x := R$ we have

$$\Delta S = 2\pi m \Delta x \,, \tag{3.17}$$

if Bekenstein bound is to be saturated for the particle. For the circumscribing radius of the Compton wavelength $\Delta x = \frac{1}{m}$ we have thus a change of entropy $\Delta S = 2\pi$. Note that this process is just imaginary, the particle is not actually dynamically moving past the screen. In the whole derivation, we assume a static configuration. Perhaps, rather than imagining the particle moving past the horizon, it is better to imagine space foliated by holographic screens, with the particle being inside some of these screens and outside others. The change of entropy then happens as we move from a screen that does not include the particle to one that does.

In [20] Verlinde's first shows how Newton's law of gravitation can emerge, and only then does he show that one can also derive Einstein equations. Here we will directly go to the relativistic setting since there is a lot of evidence pointing towards the world being relativistic, and it is quite clear how to take the Newtonian limit at every step of the derivation (we will mention the limit at a few points). An important ingredient here is that we have a static spacetime. That means by Definition A.10 that we have an asymptotically timelike Killing vector field ξ^a that is hypersurface orthogonal. Since we have a static equilibrium configuration we also have $\pounds_\xi S = \xi^a \partial_a S = 0$ and we can covariantly write (3.17) as

$$\partial_a S = -2\pi m \, n_a \,, \tag{3.18}$$

where n_a is the spatial ($\xi^a n_a = 0$) vector normal to the screen pointing in the outward direction. The minus sign comes from the fact that the entropy increases when we cross from the outside to the inside of the screen.

At this point, it is interesting to notice the following. If one, in analogy with osmosis, assumes that there is no change in the internal energy of the system and we are at equilibrium, we can use the formula (3.14) which gives us (if we assume the particle to be small and carry a small amount of entropy)

$$F = T\frac{\Delta S}{\Delta x} \xrightarrow{\Delta x \to 0} F_a = T\partial_a S \tag{3.19}$$

From this we see that in order to have some non-vanishing entropic force (we would like this to be the force of gravity) we must associate to the screen a non-zero temperature. In Jacobson's approach of the previous section, we associated to local Rindler horizons Unruh temperature and we will now do the same for our holographic screens.

To get to the screen temperature, let us first set up the stage and introduce the quantity

$$\phi := \ln|\xi|\,, \tag{3.20}$$

where here $|\xi| = \sqrt{g_{ab}\xi^a\xi^b}$[1]. The Killing vector field is defined so that at asymptotic infinity we have $\phi = 0$. This is a natural relativistic generalization of Newton's potential [23]. To see that, remember that in the Newtonian limit (weak field, weak variations of the field and slow motion) Newton's potential Φ can be defined from the time component of the metric as $g_{tt} = 1 + 2\Phi \approx e^{2\Phi}$ where in this approximation we have used the assumption of a weak field $|\Phi| \ll 1$. Time is here precisely defined by the Killing vector field through $\xi^a = \partial_t$ and thus we have $g_{tt} = g(\partial_t, \partial_t) = g(\xi,\xi) = |\xi|^2 = e^{2\phi}$. Taking the Newtonian limit of our relativistic approach is thus basically just about expanding in small ϕ and considering it to be Newton's potential. It is also easy to see that $|\xi| = e^\phi$ is the redshift factor as the proper time is for static observers given by $d\tau = \sqrt{g_{tt}}dt = |\xi|dt$ (as we have seen in the past).

We define our holographic screens as the spatial surfaces of constant Killing time and ϕ. First, consider the temperature from the point of view of our test particle. There, the Unruh temperature will be given, in analogy with Section 1.2, by

$$T_{\text{local}} = \frac{\alpha_{\text{local}}}{2\pi}\,, \tag{3.21}$$

where $\alpha_{\text{local}} = \sqrt{|a_b a^b|}$ the magnitude of four-acceleration of the particle.

To get the four-acceleration, we remember that we have a static configuration, so our particle is a static observer and those follow the integral curves of the Killing vector field, so that by definition their four-velocity is $u^a = \frac{\xi^a}{|\xi|} = e^{-\phi}\xi^a$. From this, the four-acceleration can be obtained by

$$a^b = \nabla_u u^b = u^c\nabla_c u^b = e^{-\phi}\xi^c\nabla_c(e^{-\phi}\xi^b) = e^{-2\phi}\xi^c\nabla_c\xi^b - e^{-2\phi}\xi^c\partial_c\phi\,\xi^b = e^{-2\phi}\xi^c\nabla_c\xi^b\,, \tag{3.22}$$

where we have used that $\xi^c\partial_c\phi = 0$. This is because

$$\xi^c\partial_c\phi = \pounds_\xi \ln|\xi| \propto \pounds_\xi(g_{ab}\xi^a\xi^b) = \xi^a\xi^b\pounds_\xi g_{ab} + 2g_{ab}\xi^a\pounds_\xi\xi^b = 0 \tag{3.23}$$

where $\pounds_\xi g_{ab} = 0$ is the condition for ξ being a Killing vector and $\pounds_\xi\xi^b = [\xi,\xi]^b = 0$. $\pounds_\xi g_{ab} = 0$ implies the Killing equation from which $\nabla_a\xi_b = -\nabla_b\xi_a$. It then follows that

$$\xi^c\nabla_c\xi^b = -\xi^c\nabla^b\xi_c = -\frac{1}{2}\nabla^b(\xi_c\xi^c) = -e^{2\phi}\partial^b\phi\,. \tag{3.24}$$

[1] In this section, unlike previously in this work, we reserve ξ for the abstract vector $\xi = \xi^a$ or the differential form $\xi = \xi_a$. There is a slight notation collision but which is used should be clear from the context.

Substituting this into (3.22) we finally get

$$a^b = -\partial^b \phi \,. \tag{3.25}$$

The local temperature is thus $T_{\text{local}} = \frac{1}{2\pi}|\partial\phi|$, where in the last equality we introduced the notation $|\partial\phi| = \sqrt{|\partial_a\phi\partial^a\phi|}$. In analogy with Section 1.2 we want the temperature instead as seen by an observer at the asymptotic infinity which we denote T. It is easy to see (see (2.2)) that local energy E_{local} is redshifted as $E_\infty = |\xi|E_{\text{local}}$ where E_∞ is the energy as seen by the observer at asymptotic infinity. The invariant statistical properties of the spectrum that depend on the energy-temperature ratio must remain unchanged so the temperature must redshift in the same way as energy. So we obtain

$$T = \frac{e^\phi}{2\pi}|\partial\phi| \,. \tag{3.26}$$

Now we can return to (3.19) and substitute (3.26) together with (3.18) to get

$$F_a = T\partial_a S = -me^\phi n_a|\partial\phi| = -me^\phi\partial_a\phi \,. \tag{3.27}$$

In the last equality, we noticed that $\partial^a\phi$ points in the same direction as n^a, as it is normal to the screens since these are surfaces of constant ϕ and also points outward because ϕ increases in the outward direction. But what we got is exactly the correct relativistic force that is exerted on the particle by the gravitational field, as measured by a stationary observer at the asymptotic infinity (additional redshift e^ϕ is due to energy redshift). Thus, we see that gravitational force really emerged as an entropic force. It is also easy to see that this reproduces the familiar result in the Newtonian limit as $e^\phi = 1 + \phi + O(\phi^2)$, but the term $\phi\partial_a\phi$ is already second order small, so we get (after the identification $\phi = \Phi$) just $\mathbf{F} = -m\nabla\Phi$, which is exactly Newton's law of gravity.

Let us take that a holographic screen $\mathcal{S}$ of our consideration is enclosing a mass M. Now two important assumptions of Verlinde's approach come into play. First is that the density of the number of degrees of freedom on the screen is $\frac{c^3}{G\hbar}$ so that in Planck units

$$dN = dA \,. \tag{3.28}$$

This can be maybe taken as a definition of the Newton's constant G and the density of bits on the screen maybe does not even have to be constant if we allow for nonconstant G. Notice that this differs from the Bekenstein-Hawking formula (1.36) by a factor of 1/4. The second central assumption of this approach is the assumption of equipartition between the degrees of freedom on the screen, which is expressed by the formula

$$M = \frac{1}{2}\int_{\mathcal{S}} T dN \,. \tag{3.29}$$

The holographic screens are surfaces of constant Killing time so orthogonal to ξ^a and also of constant ϕ, that is orthogonal to $\partial^a\phi$. This gives us the induced volume element on the screen as $dA = \epsilon_{abcd}\frac{\partial^a\phi}{|\partial\phi|}\frac{\xi^b}{|\xi|}\,dx^c\wedge dx^d$ where $\omega = \epsilon_{abcd}\,dx^a\wedge dx^b\wedge dx^c\wedge dx^d$ (we are taking a little bit unconventional definition without the factor 1/4!) is the spacetime volume element. Substituting into (3.29) first (3.26) and (3.29) and then using our induced area element on the screen, we have

$$M = \frac{1}{4\pi}\int_{\mathcal{S}} |\xi||\partial\phi|dA = \frac{1}{4\pi}\int_{\mathcal{S}} \partial^a\phi\,\xi^b\epsilon_{abcd}\,dx^c\wedge dx^d = \frac{1}{4\pi}\int_{\mathcal{S}} \star(d\phi\wedge\xi) \,, \tag{3.30}$$

where we used $(\star\, d\phi\wedge\xi)_{cd} = \frac{1}{2}(d\phi\wedge\xi)^{ab}\epsilon_{abcd} = \partial^{[a}\phi\,\xi^{b]}\epsilon_{abcd} = \partial^a\phi\,\xi^b\epsilon_{abcd}$ ($\star$ is the operator of the Hodge dual).

Now this can be further simplified since

$$d\phi\wedge\xi = \frac{1}{2}d\xi\,. \tag{3.31}$$

Let us prove this by introducing a form $\Omega = d\xi - 2d\phi\wedge\xi$ and proving $\Omega = 0$. For this it is enough to prove that $\iota_\xi\Omega = 0$ ($\Omega_{0i} = 0$) and $\xi\wedge\Omega = 0$ ($\Omega_{ij} = 0$). First let us use (3.24) to write

$$(\iota_\xi d\xi)_b = 2\xi^a\nabla_{[a}\xi_{b]} = 2\xi^a\nabla_a\xi_b = -(2|\xi|^2 d\phi)_b\,, \tag{3.32}$$

where we also made use of the Killing equation in the second and third equality. Now we look at

$$\iota_\xi\Omega = \iota_\xi d\xi - 2(\iota_\xi d\phi\,\xi - \iota_\xi\xi\; d\phi) = \iota_\xi d\xi + 2|\xi|^2 d\phi = 0\,. \tag{3.33}$$

The first equality is again just the Killing equation but the second one uses (3.23) and the third equality (3.32). The exterior derivative also vanishes as

$$\xi\wedge\Omega = \xi\wedge d\xi - 2|\xi|^2\xi\wedge d\phi\wedge\xi = 0\,, \tag{3.34}$$

with first term vanishing because of the ξ^a being hypersurface orthogonal which implies (A.11) and the second vanishes as we are wedging two identical forms.

Substituting (3.31) into (3.30) we have ended up with

$$M = \frac{1}{8\pi}\int_{\mathcal{S}}\star\, d\xi = \frac{1}{8\pi}\int_{\mathcal{S}}\nabla^{[a}\xi^{b]}\epsilon_{abcd}\,dx^c\wedge dx^d = \frac{1}{8\pi}\int_{\mathcal{S}}\nabla^a\xi^b\epsilon_{abcd}\,dx^c\wedge dx^d\,. \tag{3.35}$$

Now we would like to apply the generalised Stokes theorem. This means we would like to know $d\star d\xi$ but it is more convenient to first calculate $\star\, d\star d\xi$ as

$$(\star\, d\star d\xi)_d = \frac{1}{6}(d\star d\xi)^{abc}\epsilon_{abcd} = \frac{3}{6}\nabla^{[a}(\star d\xi)^{bc]}\epsilon_{abcd} = \frac{1}{2}\nabla^a\nabla_e\xi_f\epsilon^{efbc}\epsilon_{abcd} = \tag{3.36}$$

$$= 2\nabla^a\nabla_e\xi_f\delta_a^{[e}\delta_d^{f]} = 2\nabla^a\nabla_e\xi_f\delta_a^e\delta_d^f = 2\nabla^a\nabla_a\xi_d = 2R_{dc}\xi^c\,, \tag{3.37}$$

where fourth equality comes from $\epsilon^{abcd}\epsilon_{abef} = -4\delta_e^{[c}\delta_f^{d]}$, fifth is the Killing equation and the last equality we will now show. The definition of a Riemann tensor for torsionless connections basically reads $[\nabla^a,\nabla_d]\,\xi_b = \xi_c R^c{}_b{}^a{}_d$. We can contract this to get for the case of a Killing vector field

$$-\nabla^a\nabla_a\xi_d = \nabla^a\nabla_d\xi_a = \nabla^a\nabla_d\xi_a - \nabla_d\nabla^a\xi_a = \xi_c R^c{}_a{}^a{}_d = \xi_c R^c{}_d\,. \tag{3.38}$$

Here the first and second equality come from the Killing equation and the last one is the definition of the Ricci tensor. Now since in this case $p = 3$, we have $\star^2 = (-1)^{p(4-p)+1} = 1$ and thus acting with the Hodge operator on (3.36), we get $d\star d\xi = 2\star J$ where I introduced the conserved Komar current one form $J = R_{dc}\xi^c\,dx^d$ as it is defined for example in [86]. All in all we have

$$M = \frac{1}{4\pi}\int_{\mathcal{V}}\star J = \frac{1}{4\pi}\int_{\mathcal{V}}R_{dc}\xi^c\,dx^d\epsilon^d{}_{abe}\,dx^a\wedge dx^b\wedge dx^e = \frac{1}{4\pi}\int_{\mathcal{V}}R_{ab}\frac{\xi^a\xi^b}{|\xi|}\,dV\,, \tag{3.39}$$

where I have used that since $\mathcal{V}\subset\Sigma$ where Σ is the space-like hypersurface orthogonal to ξ^a, we have that $\epsilon^d{}_{abe}\,dx^a\wedge dx^b\wedge dx^e = \frac{\xi^d}{|\xi|}\frac{\xi^c}{|\xi|}\epsilon_{cabe}\,dx^a\wedge dx^b\wedge dx^e = \frac{\xi^d}{|\xi|}\,dV$ where $dV = \frac{\xi^c}{|\xi|}\epsilon_{cabe}\,dx^a\wedge dx^b\wedge dx^e$ is the volume form on the hypersurface Σ.

Equation (3.39) connects the mass or energy enclosed inside the holographic screen to the emergent geometry of the region. We have thus already captured the essence of Einstein equations, but let us make it more precise. Looking at M from the other side, we would expect to get it as some integral involving the energy-momentum tensor. The precise form is basically a matter of our definition, but one can get the specific combination by looking at the expected properties like conservation laws in a similar way to that at the end of the last section. Based on this, we could argue [20] that the mass enclosed inside a volume $\mathcal{V}$ at constant Killing time slice could be defined as an integral of the current $J_M = 8\pi \left(T_{ab} - \frac{1}{2}T g_{ab}\right) \xi^a \, dx^b$, specifically

$$M := \frac{1}{4\pi} \int_{\mathcal{V}} \star J_M = 2 \int_{\mathcal{V}} \left(T_{ab} - \frac{1}{2}T g_{ab}\right) \frac{\xi^a \xi^b}{|\xi|} \, dV \,. \tag{3.40}$$

Connecting (3.40) and (3.39) one thus obtains

$$8\pi \int_{\mathcal{V}} \left(T_{ab} - \frac{1}{2}T g_{ab}\right) \frac{\xi^a \xi^b}{|\xi|} \, dV = \int_{\mathcal{V}} R_{ab} \frac{\xi^a \xi^b}{|\xi|} \, dV \,. \tag{3.41}$$

We have thus obtained the integrated form of Einstein equations. At the end of his paper, Verlinde argues [20] that from this one could get the full equations possible through some reasoning similar to how we got from (3.7) the Einstein equations in the last section in Jacobson's approach. But Verlinde himself admits that his argument is quite sketchy and would benefit from being precisely formulated, which I, however, have not found elsewhere.

Since the end of the relativistic derivation is admittedly a little bit hand-wavy, let us quickly mention the Newtonian limit of the derivation, which is simpler and thus might be clearer. In this limit, the temperature (3.26) reduces to

$$T = \frac{|\nabla\Phi|}{2\pi} \,. \tag{3.42}$$

The equipartition equation (3.29) is in this setting actually identical, and we may substitute (3.42) to obtain

$$M = \frac{1}{4\pi} \int_{\mathcal{S}} |\nabla\Phi| \, dA = \frac{1}{4\pi} \int_{\mathcal{S}} \nabla\Phi \cdot dA \,, \tag{3.43}$$

where in the second equality we used that $\nabla\phi$ points in the direction of the normal of the area element of the screen $\mathcal{S}$. If (3.43) is to hold for an arbitrary screen $\mathcal{S}$ given by an equipotential surface, the Poisson equation

$$\nabla^2\Phi(\mathbf{r}) = 4\pi\rho(\mathbf{r}) \tag{3.44}$$

follows [20].

Let us also mention that ϕ can be seen as a coarse-graining variable. To see this, let us assume that since a particle crossing a holographic screen merges with the degrees of freedom on the screen, it is made of the same kind of fundamental degrees of freedom for which we assumed equipartition before. Let us say that a particle of mass m is constituted from n such degrees of freedom, then we have

$$m = \frac{1}{2} n \, T \,. \tag{3.45}$$

Substituting this into (3.18) and also using (3.26) leads to

$$2 \, ds = \frac{2 \, dS}{n} = -e^{\phi} |\partial\phi| n_a \, dx^a = -e^{\phi} \, d\phi \approx -d\Phi \tag{3.46}$$

where $dS = \partial_a S\, dx^a$, $d\phi = \partial_a \phi\, dx^a$, the last approximation holds in the weak field limit and ds denotes the infinitesimal increment of entropy per degree of freedom. We thus see that the increase in entropy per degree of freedom corresponds to half the decrease in Newton's potential. Thus, Newton's potential plays the role of a coarse-graining variable. This can be understood by looking, for example, at the Schwarzschild solution. There, the equipotential surfaces are spheres which decrease in radius as we get closer to the source, and that is also the direction of decreasing potential. So, as the intensity of the potential increases (the potential itself decreases as it is negative) the radius of the holographic screen decreases, so the area of the screen shrinks. We have assumed that the density of bits is constant, so as we delve deeper into the gravitational well, we have less and less bits to encode the same amount of information (all the screens include all of the mass) so one can maybe imagine sort of a spin coarse-graining procedure as the screens get smaller. The smallest value of the potential one can get is at a black hole event horizon, where we have $2\phi = -1$, so the maximum coarse-graining happens at black hole event horizons.

One can draw an analogy with AdS/CFT from Section 2.3. There we have seen that we have also sort of holographic screens which foliate the space and can be understood as moving the CFT from the conformal boundary to some AdS radius. These can be even defined based on the same principle, that is as surfaces of constant $\phi = \ln|\xi| = \frac{1}{2}\ln g_{tt}$ and t where t is the Killing time. For the Poincaré patch with metric (2.15) we have $\phi = \ln\frac{L}{z}$ where z is the radial coordinate so surfaces of constant ϕ are surfaces of constant z which, as we discussed, are at constant time just rescaled $\mathbb{R}^3$. For the global AdS one has instead the metric (2.16) and $\phi = \frac{1}{2}\ln\left(1+\frac{r^2}{L^2}\right)$ so again surfaces of constant ϕ are surfaces of constant radial coordinate r which are at a given time 3-spheres of different radii. The coarse-graining effect of Verlinde has also an analogous feature of AdS/CFT. We have seen that the radial coordinate, which we have just discussed, ϕ is a monotonic function of, can be understood as emerging from the RG flow. We also have in the global AdS that as ϕ decreases, r decreases and the holographic screens are spheres of increasingly smaller radii which correspond to a larger and larger cut-off in the boundary CFT and thus increasing coarse-graining as one would expect in Verlinde's approach.

To summarise we have seen how Verlinde let gravity emerge as an entropic force. We have shown the full derivation of Einstein equations pointing out how it reduces in the Newtonian limit. As we have seen and also others [14] have noticed there are some gaps in the full derivation of general relativity but the Newtonian derivation is quite plausible under appropriate assumptions. However, many assumptions are required on the plausibility of which it has been commented on in [20] and we left out here most of this discussion. With that in mind I am personally sceptical about the assumptions being satisfied though it has also been pointed out that some are unnecessary [14]. Some of the main assumptions mentioned include:

- There is a change of entropy in the emergent direction associated with holographic screens containing different number of fundamental particles.
- Bekenstein bound is saturated for fundamental particles.
- The holographic screens have temperature in the form of Unruh temperature.
- The are fundamental degrees of freedom uniformly distributed over the holographic screens and their number is equal to the area in Planck units.
- These degrees of freedom obey the law of equipartition.
- The spacetime is static.

We have seen that an important ingredient in Jacobson's approach of the last section was sort of a local saturation of Bousso's bound. Here we instead see as one of the assumptions saturation of the Bekenstein bound for particles. And based on Section 2.1 Bekenstein bound appears to me less fundamental. The approach of thermodynamic gravity also seems to me better formulated with less controversial assumptions and requires only local thermodynamic equilibrium for the spacetime (whatever that means) not specifically staticity. On the other hand, Verlinde's approach has a nice similarity to the AdS/CFT correspondence. Also, the microscopic structure and the process of emergence of spacetime seemed here to me more vivid.

Similarly as we have seen for the approach of the last section, also in Verlinde's entropic gravity, the Friedmann equations of the FLRW universe can be derived in this manner [104]. There also have been many further works extending and generalising these ideas, as for example [105] where it has been discussed how this form of emergent gravity can differ from the predictions of general relativity. One example of a related experimental work is [106] where authors argue based on experiments with ultracold neutrons in the gravitational field of Earth against Verlinde's approach. There has been an interesting work responding to this and suggesting these supposed issues can be resolved by the use of Tsallis non-additivity entropy [107] which plays an important role in the next section.

3.3 Use of generalised entropies in emergent gravity

In previous sections we have seen different theories that treat gravity as an emergent interaction. We have also seen that this opens space for different deviations from standard general relativity. Jacobson, for example, considered non-equilibrium effects [97] in his approach, which we have seen in Section 3.1. On the other hand, in the paper [105] Verlinde argues that de Sitter space contains both an area-law entropy (as we have seen in Section 3.2) and a volume-law contribution due to long-range entanglement in the underlying microscopic degrees of freedom, which leads to an 'elastic' response of spacetime to matter. In this section, we will see how generalized entropies can be useful both in established results in emergent gravity and in potential modifications that may come from their use. We will focus on the latter, with the main subject of this section being an example of generalized gravity in the form of a deformation of the Friedmann equation based on Tsallis entropy, as introduced in the paper [22].

Already in Newtonian gravity, systems of gravitating particles have been found to have statistical peculiarities due to the long-range, universal nature of the interaction [108]. It has been found that the partition function diverges both at large distances if the system is not confined (due to the non-integrability of the Newtonian potential) and at short distances (due to the singularity of the potential at zero separation). This was already highlighted by Gibbs [109]. Other problems include negative specific heat (we have seen this when we looked at the specific heat of the Schwarzschild black hole in equation (1.35)), inequivalence of ensembles, absence of equilibrium in the standard sense (gravitational systems do not generally reach a global thermodynamic equilibrium without external confinement, and instead can evolve toward collapse or evaporation), and non-extensivity of energy and entropy [108].

The non-additivity and non-extensivity of entropy is often found in the presence of long-range interactions [110, 111]. This is connected to the breakdown of ergodicity in such systems [112], which can be clearly seen in the holographic principle (see Chapter 2) where most of the state space of a gravitating system in general relativity results in a black hole, as highlighted by equation (2.14). This is also related to the black hole area law, where the entropy of a black hole scales as the area rather than the volume, as one would normally expect. In the paper [113]

Tsallis and Cirto argue that if a black hole is to be considered a 3D system (which is not at all obvious, precisely because of the holographic principle, which suggests that the true degrees of freedom could live only on the lower-dimensional horizon as discussed in detail in Section 2.2 and throughout this work), then one should not use the Boltzmann-Gibbs entropy but some generalized entropy functional. The famous generalized entropy functional introduced by Tsallis [114] has, for example, found use in such settings involving the breakdown of extensivity in the presence of long-range interactions and the failure of ergodicity [115].

A natural extension of the use of generalized entropies in gravitation is to replace the Bekenstein–Hawking form S = A/4 with a power-law expression [22]

$$S = \gamma A^{\beta}\,, \tag{3.47}$$

where β is a scaling exponent and γ a constant. Such a modification can be implemented directly in the first law of thermodynamics applied to horizons, leading to corrections to general relativity. A classic example is the derivation of modified Friedmann equations from this entropy form, as carried out in [22], which recovers the standard Friedmann equations when $\beta = 1$. This is using the techniques developed in the derivation of the standard Friedmann equation from the first law with area law for the apparent horizon [88–91]. While Tsallis and Cirto [113] emphasized the case $\beta = 3/2$ for black holes in three spatial dimensions, they did not exclude other values, and later works have explored the consequences of leaving β free. This broader viewpoint is shared by [22], where the goal is to capture generic deviations from the area law, and it is also supported by deformations arising in Barrow entropy [116, 117], which yield a similar power-law modification. An alternative approach is taken in [118], where the exponent β is tied to higher-curvature gravity and second-law considerations, leading to an 'effective' (3.47) β possibly differing from 3/2. In what follows, we focus on the [22] framework as a starting point for obtaining Friedmann equations from the first law with such a generalized entropy.

In order to apply this procedure in a cosmological setting, we consider a spatially homogeneous and isotropic universe described by the FLRW metric

$$g = h - R^2(t)d\Omega^2\,, \quad \text{where} \quad h = dt^2 - a^2(t)\frac{dr^2}{1-kr^2}\,. \tag{3.48}$$

Here $a(t)$ is the scale factor, $k \in \{-1, 0, 1\}$ denotes the spatial curvature, r is the commoving radial coordinate, and $d\Omega^2$ is the metric of $\mathbb{S}^2$. The quantity $R(t) = a(t)r$ is called the areal radius. It is defined so that a sphere at commoving radius r has proper area $A(t) = 4\pi R^2(t)$, which is the reason for the name 'areal'.

As in Section 3.1 we have seen how Einstein equations emerge from the Clausius relation applied to local Rindler horizons. Here we will see how the Friedmann equation emerges from the first law of thermodynamics in the form

$$dE = T\,dS - w\,dV \tag{3.49}$$

applied to the apparent horizon. We will see how these quantities are defined throughout. These definitions are based on [88].

First we have to find the location of the apparent horizon. Because of the spherical symmetry, we see that it will be a surface of constant r (a sphere). From Definition A.12 it is clear that this is the surface where one of the expansions vanishes. Now again comes in handy the formula (2.10) from which we see that

$$\theta_{\pm} = \frac{1}{A}\partial_{\pm}A = \frac{2}{R}\partial_{\pm}R\,, \tag{3.50}$$

where ∂_+ and ∂_- are the future-directed outgoing and ingoing light directions forming the basis of the subspace orthogonal to the spheres of constant r. In these coordinates, we can also write

$$-h = dx^+ \otimes dx^- + dx^- \otimes dx^+ \,. \tag{3.51}$$

Now it is useful to realize that

$$h^{ab}\partial_a R \partial_b R = h^{+-}\partial_+ R \partial_- R + h^{-+}\partial_- R \partial_+ R = -2\partial_+ R \partial_- R = -\frac{R^2}{2}\theta_+\theta_- \,, \tag{3.52}$$

so if we want to find a surface where one of the expansions vanishes it is enough to find surface where $h^{ab}\partial_a R \partial_b R = 0$. As we have $\partial_t R = \dot{a}r = Har = HR$ (here we introduced the Hubble parameter $H = \frac{\dot{a}}{a}$) and $\partial_r R = a$, we get

$$h^{ab}\partial_a R \partial_b R = g^{tt}(\partial_t R)^2 + g^{rr}(\partial_r R)^2 = -H^2R^2 + 1 - kr^2 = -H^2R^2 + 1 - \frac{kR^2}{a^2} = 0 \,. \tag{3.53}$$

This is easily solved to obtain

$$R_A = \frac{1}{\sqrt{H^2 + \frac{k}{a^2}}} \,, \tag{3.54}$$

for the areal radius at which the apparent horizon is located.

It has been shown that even in the dynamical FLRW spacetime still holds the formula

$$T = \frac{\kappa}{2\pi} \tag{3.55}$$

for the temperature of the Hawking radiation coming from the presence of the apparent horizon [91]. To get to the temperature we will thus need the surface gravity. In Section 1.2 we have used Definition A.9 but since we now do not have the timelike Killing vector field, the apparent horizon cannot be a Killing horizon. However, there is a generalisation of the concept of timelike Killing vector field for spherically symmetric dynamical spacetimes which is the Kodama vector field [119]. It is defined as

$$K_a = (\star_h \, dR)_a = \epsilon_{ab}\partial^b R \,, \tag{3.56}$$

where R is the radial coordinate of the space which is in our case exactly the areal radius which we used throughout. One can then define a Kodama horizon in the same way as a surface where K^a turns null which here exactly coincides with the apparent horizon as can be seen for example from

$$K^a K_a = \epsilon_{ab}\epsilon^{ac}\partial^b R \partial_c R = -\delta^c_b \partial^b R \partial_c R = -h^{ab}\partial_a R \partial_b R \,. \tag{3.57}$$

For Killing vector field the surface gravity κ of a Killing horizon $\mathcal{K}$ can be defined as the proportionality constant in $\xi^a \nabla_a \xi_b \overset{\mathcal{K}}{=} \kappa \xi_b$. Because of the Killing equation $\nabla_{(a}\xi_{b)} = 0$, we have also that $\xi^a \nabla_a \xi_b = \xi^a \nabla_{[a}\xi_{b]} = (\frac{1}{2}\iota_\xi d\xi)_b$. Here we do not have a Killing equation, so these forms will not be equivalent and the one that is generalized to Kodama vector fields is the latter. So the surface gravity is defined for the apparent horizon $\mathcal{A}$ by

$$\iota_K dK \overset{\mathcal{A}}{=} 2\kappa K \,. \tag{3.58}$$

This definition can be actually shown [88] to be equivalent to the formula $\kappa = \frac{1}{2}\Box_h R\big|_{\mathcal{A}} = \frac{1}{2}h^{ab}\partial_a\partial_b R\big|_{\mathcal{A}}$. Using this, it is straightforward (but a little tedious) to show that

$$\kappa = -\frac{1}{R_A}\left(1 - \frac{\dot{R}_A}{2HR_A}\right) \,. \tag{3.59}$$

As usual, we assume the matter and energy content of the Universe to be a commoving perfect fluid; that is, the energy-momentum tensor reads

$$T^{ab} = (\rho + p)u^a u^b - p g^{ab}\,, \tag{3.60}$$

where the four-velocity is in our commoving coordinates $u^\mu = (1, 0, 0, 0)$, ρ is the energy density and p is the pressure. The work density done by the apparent horizon is only in the orthogonal direction, which motivates defining it as

$$w = -\frac{1}{2} h^{ab} T_{ab}\,, \tag{3.61}$$

as has been done for example in [88]. From this, we readily obtain

$$w = -\frac{1}{2}(\rho + p) h_{ab} u^a u^b + \frac{1}{2} p h^{ab} g_{ab} = -\frac{\rho + p}{2} u^a u_a + \frac{1}{2} p \delta^a_a = \frac{p - \rho}{2}\,. \tag{3.62}$$

It is interesting to notice that for a pure de Sitter space where $\rho = -p$, (3.49) reduces to the usual form of the first law of thermodynamics $dE = T\,dS - pdV$. From energy momentum conservation $\nabla^a T_{ab} = 0$ in the FLRW spacetime, we also have the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0\,. \tag{3.63}$$

The Kodama vector field gives us additionally the two currents K_a and $J_a = T_{ab} K^b$ which are conserved as $d \star K = 0$ and $d \star J = 0$. We may thus define associated charges on a spatial hypersurface Σ as

$$E = \int_\Sigma \star J = \int_\Sigma u_a J^a\, dV \quad \text{and} \quad V = \int_\Sigma \star K = \int_\Sigma u_a K^a\, dV\,, \tag{3.64}$$

where dV is the volume form on Σ and u^a is the timelike unit vector orthogonal to Σ. These then turn out to be the correct notions of energy and volume that should be used in (3.49) if we choose Σ to be the region inside the apparent horizon [88, 90]. As we have $K = \star_h dR$ where h is the metric on subspace orthogonal to $\partial\Sigma$ and $\partial\Sigma$ is a sphere of radius R_A, it is not that hard to prove that $V = \frac{4}{3}\pi R_A^3$, because of which is V called the areal volume. Also, since $J_a u^a = T_{ab} u^a K^b = \rho\, u_b K^b$ we have

$$E = \int_\Sigma u_a J^a\, dV = \rho \int_\Sigma u_b K^b\, dV = \rho V\,, \tag{3.65}$$

where in the second equality we also used that ρ is constant on surfaces of constant time in the commoving coordinates. E is exactly what is called the Misner-Sharp energy [120].

Now we are ready to substitute everything into (3.49). First we have

$$\frac{dE + w\, dV}{3V} = H(\rho + w)\frac{dV}{3HV} + \frac{1}{3}\, d\rho = -H(\rho + p)\left(1 - \frac{\dot{V}}{6HV}\right) dt\,, \tag{3.66}$$

where in the first equality we substituted (3.65), in the second equality we then used (3.62) and the continuity equation (3.63) in the form $d\rho = -3H(\rho + p)\, dt$. As we have $\frac{\dot{V}}{6HV} = \frac{\dot{R}_A}{2HR_A}$, we can substitute this into (3.66) and all of that to (3.49) to obtain

$$-H(\rho + p)\left(1 - \frac{\dot{R}_A}{2HR_A}\right) dt = \frac{TdS}{3V} = -\frac{\gamma\beta(4\pi R_A^2)^{\beta-1}}{\pi R_A^3}\left(1 - \frac{\dot{R}_A}{2HR_A}\right) dR_A \tag{3.67}$$

where in the first equality we used (3.55) with (3.59) and also from (3.47)

$$dS = \gamma\beta A^{\beta-1}dA = 8\pi R_A\gamma\beta\,(4\pi R_A^2)^{\beta-1}\,dR_A\,. \tag{3.68}$$

Finally, we again use the continuity equation $d\rho = -3H(\rho+p)\,dt$ to get

$$-\frac{(4\pi R_A^2)^{\beta-1}}{R_A^3}\,dR_A = \frac{\pi}{3\gamma\beta}\,d\rho\,. \tag{3.69}$$

We integrate this

$$-(4\pi)^{\beta-1}\int R_A^{2\beta-5}\,dR_A = \frac{\pi}{3\gamma\beta}\int d\rho \tag{3.70}$$

and setting the constant of integration to 0, end up with

$$R_A^{2\beta-4} = \frac{2\pi(2-\beta)}{3\gamma\beta}(4\pi)^{1-\beta}\rho\,. \tag{3.71}$$

Using (3.54) we finish with the modified Friedmann equation

$$\boxed{\left(H^2+\frac{k}{a^2}\right)^{2-\beta} = \frac{8\pi}{3}\rho\,,} \tag{3.72}$$

provided that we set

$$\gamma := \frac{2-\beta}{4\beta}(4\pi)^{1-\beta}\,. \tag{3.73}$$

In the limiting case where $\beta = 1$, we recover the standard Friedmann equation in Einstein gravity. It is also clear that we always have $\beta < 2$, which puts an upper bound on the non-additive entropy parameter β.

Other works also looked at how this relates to other approaches to modified gravity. Wald introduced [121] viewing black hole entropy as Noether's charge of a black hole horizon, which allows one to connect the Lagrangian of a theory to a given entropy functional. For a Lagrangian of a generic gravitational theory of the form $\mathcal{L}(g_{ab}, R_{abcd}, \nabla_a R_{bcde})$ one gets the black hole entropy as

$$S = -2\pi\int_{\mathcal{H}} \nabla_a\chi_b\,\nabla_c\chi_d\frac{\delta L}{\delta R_{abcd}}\,dA\,, \tag{3.74}$$

where χ is the Killing field generating the black hole horizon and $\mathcal{H}$ is the bifurcation surface of the horizon with an area element dA. Using this, the paper [122] looked at what would be the theory corresponding to the entropy functional (3.47). The work [123], on the other hand, connected this generalised entropy to the generalised uncertainty relations which were the subject of the previous work [15].

Now we come to uses of generalized entropies in emergent gravity beyond deformations. In the study of entanglement entropy in QFT, an important tool is the so-called Rényi entanglement entropy [124, 125], which for a density matrix ρ is defined as

$$S_q(\rho) = \frac{1}{1-q}\ln\mathrm{Tr}(\rho^q) \tag{3.75}$$

for a non-extensivity parameter $q \geq 0$ (for $q = 1$ it is defined via the corresponding limit), and which reduces in the limit $q \to 1$ to the standard von Neumann entropy $S(\rho) = -\mathrm{Tr}(\rho\ln\rho)$. The

Rényi entanglement entropy plays a central role in the replica trick. Instead of attempting to compute directly the von Neumann entropy, which involves $\ln \rho$ and is often difficult to evaluate, one first computes Rényi entropy for integer q by evaluating the partition function on a q-sheeted replica manifold, then analytically continues the result to real q and takes the limit $q \to 1$. In QFT this has traditionally been a calculational device without direct physical interpretation. In AdS/CFT, however, it has found a geometric realization in the bulk [126, 127] (see Section 2.3 for a discussion around AdS/CFT), where for integer q it is dual to the area of a codimension-two cosmic brane with a specific tension inserted in the bulk spacetime [128].

Conclusion

This work has primarily served as a review intended to lay the foundation for my forthcoming master's thesis. While the main emphasis has been on summarizing existing approaches, I have sought to contribute a distinct perspective by refining and, in some cases, modifying certain derivations. For instance, whereas the derivation of Hawking radiation in Section 1.2 is commonly presented in Schwarzschild coordinates, I have carried it out in Kruskal-Szekeres coordinates, which offers a more comprehensive treatment—even if similar approaches have likely been explored elsewhere. I have also endeavoured to interconnect the various topics so that they form a coherent and continuous narrative. The work concludes with a brief discussion of potential applications of generalized entropy functionals in the context of emergent gravity. An appendix provides supplementary derivations and clarifications that are needed in the main part of the text.

In my prospective master's thesis, I aim to investigate the use of generalized entanglement entropies within the framework of AdS/CFT. In particular, I plan to extend the discussion of the well-established applications of Rényi entropies and to explore a possible interpretation of the generalized Tsallis entanglement entropy, defined for a density matrix ρ and a non-extensive parameter $q \geq 0$ (for $q = 1$ it is defined via the corresponding limit) as

$$S_q(\rho) \;=\; \frac{\mathrm{Tr}(\rho^q) - 1}{1 - q}\,, \tag{3.76}$$

which reduces to the von Neumann entropy in the limit $q \to 1$.

There are basically two possible research directions I am considering to pursue. One possible direction would be to focus specifically on the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, which offers a particularly tractable setting for exploring the duality. This is due to the fact that AdS quantum gravity in 2+1 dimensions is an exactly solvable system [61], and a great deal is known about 1+1-dimensional conformal field theories [62]. More importantly, it has been shown in Refs. [125, 129] that, in this setup, the Tsallis entanglement entropy becomes extensive for a specific value of the non-additive parameter, namely for

$$q_{\mathrm{ext}} = \frac{\sqrt{9 + c^2} - 3}{c} \xrightarrow{c \to \infty} 1\,, \tag{3.77}$$

where c is the central charge of the CFT. In $\mathrm{AdS}_3/\mathrm{CFT}_2$ the central charge is given by

$$c = 6\,(N_1 N_5 + 1) \tag{3.78}$$

where N_1 and N_5 are the numbers of D1- and D5-branes [59]. In what we called in Section 2.3 the classical limit, the number of branes becomes infinite and thus the central charge becomes infinite. Therefore, the classical limit of the Tsallis entropy is the von Neumann entropy. One could then try to investigate whether there is a meaningful interpretation at finite brane number for the entanglement Tsallis entropy (3.76). One does not have to restrict attention only to the values q_{ext}, but this might serve as a guiding observation. A source of hope for this approach is that we know that for integer q there exists a gravitational dual for the Rényi entropy, and the two entropies are strict monotonic functions of each other for fixed q, so the existence of a gravitational dual for the latter might suggest one for the former as well.

The second possible research direction, which I have not yet explored in detail, would be to examine the potential applicability of Tsallis entropy in the context of the SYK model, discussed in Section 2.3. The SYK model describes a system of N fermions with random all-to-all interactions and an emergent conformal symmetry in the infrared regime. A conceptually similar scenario—namely, a system of N probabilistically correlated subsystems exhibiting emergent asymptotic scale invariance—has been investigated in [130]. In that work, the Tsallis entropy with $q \neq 1$ was shown to coincide with the entropy satisfying the standard Clausius-like prescriptions of classical thermodynamics. This suggests that it may be worthwhile to explore whether a genuine correspondence exists between these two settings.

Appendix A

Important concepts of general relativity

Let us state some technical terms and important theorems of general relativity used in our work. This is both to be comprehensive for the reader's convenience and to clarify their exact formulation, which may differ across various sources. We stress again that we do not strive to reach the standards of mathematical rigour in definitions and theorems. We simply aim to make the discussion more precise.

A.1 Causal structure

First, we define terms regarding causality in a spacetime. These are all relativistically invariant and often help us state the discussion in an invariant manner. Here, we follow [86] unless stated otherwise.

Before we begin it is necessary to have a concept of both past and future. Locally, defining these concepts is straightforward using the future and past light cones, but globally, it is less so. To address this, we introduce the following definitions [32].

Definition A.1. *Let us have spacetime (M, g_{ab}). We define a* ***causal vector*** *as a timelike or null vector and a* ***causal curve*** *to be a curve with an everywhere causal tangent vector.*

We say (M, g_{ab}) is ***time-orientable*** *if it admits a causal vector field T^a. A causal vector at point p in an orientable spacetime is then said to be a* ***future-directed vector*** *if it lies in the same light cone as $T^a(p)$ and a* ***past-directed vector*** *otherwise. Finally, we define a* ***future/past-directed curve*** *as a curve that has everywhere future/past-directed tangent vector.*

Since we are not being rigorous, throughout this work, we will not explicitly mention whether the employed spacetime needs to be time-orientable or not. If necessary, it will be implicitly assumed. However, it should be evident from the terms used when time-orientability is required, which is the case almost everywhere.

Having a notion of past and future we can for any subset of spacetime define the following:

Definition A.2. *Let us have spacetime (M, g_{ab}) and a set $S \subset M$. We define*

- *the* ***causal future/past*** *$J^{+/-}(S)$ of S as the set of points that can be reached from S by following a future/past-directed causal curve.*
- *the* ***chronological future/past*** *$I^{+/-}(S)$ of S as the set of points that can be reached from S by following a future/past-directed timelike curve.*

- *the set S to be* ***achronal*** *if no two points in S can be connected by a timelike curve.*

The intuition is, for example, that the causal future/past of S is the portion of spacetime that could have been/could be affected by S. Note that $\forall S \subset M$: $I^{+/-}(S) \subset J^{+/-}(S)$.

Specifically for an achronal subset, we have:

Definition A.3. *Let us have spacetime (M, g_{ab}) and an achronal set $S \subset M$. We define*

- *the* ***future/past domain of dependence*** *$D^{+/-}(S)$ of S as the set of points p for which every future/past directed inextensible causal curve through p intersects S. The set $D(S) := D^+(S) \cup D^-(S)$ is simply called the* ***domain of dependence*** *of S.*

- *the* ***future/past Cauchy horizon*** *$H^{+/-}(S) := \partial D^{+/-}(S)$.*

- *the set S to be a* ***partial Cauchy surface*** *if it is also closed hypersurface with no edge.*

- *the set S to be a* ***Cauchy surface*** *if it is a partial Cauchy surface and $D(S) = M$.*

We say that (M, g_{ab}) is ***globally hyperbolic*** *if there exists a Cauchy surface in M.*

Here $D(S)$ can be understood as the region which can be predicted solely by knowledge of S. A Cauchy surface is then a surface on which one usually specifies data for the initial value problem at hand.

A.2 Energy conditions

Generally, any metric can solve the Einstein equations for some energy-momentum tensor T_{ab}. To somewhat restrict the possibilities by requiring the sources to be in some sense realistic, without specifying the exact form of T_{ab}, we use the energy conditions. These are invariant statements about the form of T_{ab}. Here, we define the ones used in our work, following again [86].

Definition A.4. *Let us have spacetime (M, g_{ab}) with an energy-momentum tensor T_{ab}. We define*

- *the* ***null energy condition*** *as the requirement that $\forall$ null vectors $k^a : T_{ab}k^a k^b \geq 0$.*

- *the* ***weak energy condition*** *as the requirement that $\forall$ timelike vectors $u^a : T_{ab}u^a u^b \geq 0$.*

- *the* ***dominant energy condition*** *as the requirement that weak energy condition holds and $\forall$ future directed timelike vectors $u^b : T_{ab}u^b$ is a non-spacelike future directed vector.*

We have ordered the energy conditions in descending order of strength. That is: dominant energy condition $\implies$ weak energy condition $\implies$ null energy condition. The physical interpretation is that for the weak condition $T_{ab}u^a u^b$ represents the energy density as measured by an observer with four-velocity u^a and for the dominant condition $T_{ab}u^b$ is the measured energy-momentum four-current density of matter. Dominant energy condition can be then thought of as saying that the energy flow cannot travel faster than the speed of light. This can be made more precise as is discussed in [28].

On one hand ordinary classical matter usually satisfies even the dominant energy condition, on the other hand, quantum fields can generally violate any of these conditions.

A.3 Null geodesic congruences

In this section we follow mainly the treatment of [23, 86]. Let us again briefly present some results and definitions related to null geodesic congruences. This topic is very important for example in the proofs of the singularity theorems [23] and in our work for instance in the Subsection 2.1 and the Section 3.1. It also appears indirectly through the definition of the trapped surface, which is given in the next Section A.4.

The basic definition is:

Definition A.5. *Let us have a spacetime (M, g_{ab}) with an open set $\mathcal{U} \subset M$. A* ***null geodesic congruence*** *in $\mathcal{U}$ is a set of affinely parameterized geodesics $\mathcal{C}$ such that $\forall p \in \mathcal{U} : \exists! \gamma \in \mathcal{C} : p \in Im(\gamma)$ and the tangent vectors to the curves of $\mathcal{C}$ form an everywhere null vector field k^a.*

As one can see a congruence is basically a generalization of the notion of one parameter family of curves.

To study the evolution of nearby geodesics, we consider a distribution of vectors normal to k^a and study its failure to be parallel transported. First, there is some degeneracy in the normal subspace if we define it as $D(p) = \{v^a \in T_pM \,|\, k^a(p)v_a = 0\}$. This is because null vectors are normal to themselves. To solve this we instead consider at each point the factor space $\hat{D}(p) = D(p)/\mathrm{Span}\{k^a(p)\}$.

We also want to define some tensors on $\hat{D}(p)$. General spacetime tensor $T^{a_1 \dots a_k}{}_{b_1 \dots b_l}(p)$ can be made into a tensor on $\hat{D}(p)$ if it respects the factorization. That is, any contraction of it with $k^a(p)$ gives a 0 tensor on $D(p)$. We then denote the corresponding tensor on $\hat{D}(p)$ as $\hat{T}^{a_1 \dots a_k}{}_{b_1 \dots b_l}(p)$. This is true for example for the metric tensor g_{ab}.

Another important tensor that is compatible with the factorization is

$$B_{ab} := \nabla_a k_b \,. \tag{A.1}$$

It turns out that this tensor encodes everything we care about. Later for the Theorem A.7 we consider the vector field k^a to be hypersurface orthogonal. In this case B_{ab} is exactly the extrinsic curvature (as defined in [86]) of the corresponding family of hypersurfaces. We do not lose any information by considering only $\hat{B}_{ab}$, and we can gain further insight by decomposing $\hat{B}_{ab}$ into its trace, antisymmetric part and symmetric trace-free part. This is done using the metric $\hat{g}_{ab}$ and leads to the following definition

Definition A.6. *Let us have a null geodesic congruence $\mathcal{C}$ with the associated tangent vector field k^a. We define the* ***expansion*** *θ of $\mathcal{C}$ as $\theta := \nabla_a k^a = \hat{g}^{ab}\hat{B}_{ab}$, the* ***shear*** *of $\mathcal{C}$ as $\hat{\sigma}_{ab} := \hat{B}_{(ab)} - \frac{1}{2}\theta\hat{g}_{ab}$ and the* ***rotation*** *of $\mathcal{C}$ as $\hat{\omega}_{ab} := \hat{B}_{[ab]}$.*

Therefore, we have

$$\hat{B}_{ab} = \hat{\sigma}_{ab} + \hat{\omega}_{ab} + \frac{1}{2}\theta\hat{g}_{ab} \,. \tag{A.2}$$

In the Definition A.6 we already pointed out that the expansion θ is unaffected by the factorization, unlike shear and rotation, and it is a scalar, so it is easy to work with. The physical effect of θ, $\hat{\sigma}_{ab}$ and $\hat{\omega}_{ab}$ on an infinitesimal neighbourhood of a given point then really matches their names and additional discussion about the interpretation of θ can be found in the Subsection 2.1.3.

A simple application of the equation of geodesic deviation yields the evolution of B_{ab}. The equation for the expansion θ is the famous **Raychaudhuri's equation**, and it reads

$$\boxed{\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}_{ab}\hat{\sigma}^{ab} + \hat{\omega}_{ab}\hat{\omega}^{ab} - R_{ab}k^a k^b\,.} \tag{A.3}$$

Let us reveal some important consequences of the Raychaudhuri's equation. First, notice that since $\hat{\sigma}_{ab}$ and $\hat{\omega}_{ab}$ are both 'purely spatial' tensors, we have

$$\hat{\sigma}_{ab}\hat{\sigma}^{ab} \geq 0 \quad \text{and} \quad \hat{\omega}_{ab}\hat{\omega}^{ab} \geq 0\,. \tag{A.4}$$

It also turns out that [86]

$$k^a \text{ is hypersurface orthogonal } \iff \hat{\omega}_{ab} = 0\,. \tag{A.5}$$

The Einstein equations with $\Lambda = 0$ than imply

$$R_{ab}k^a k^b = 8\pi\left(T_{ab} - \frac{1}{2}T g_{ab}\right)k^a k^b = 8\pi T_{ab}k^a k^b\,. \tag{A.6}$$

So assuming that k^a is hypersurface orthogonal and the Einstein equations (with $\Lambda = 0$) along with the null energy condition hold, the last three terms in the Raychaudhuri's equation (A.3) are non-positive, and we get

$$\frac{d\theta}{d\lambda} \leq -\frac{1}{2}\theta^2\,. \tag{A.7}$$

One can also further estimate this to get simply

$$\frac{d\theta}{d\lambda} \leq 0\,, \tag{A.8}$$

or one can further integrate this inequality to obtain

$$\frac{1}{\theta(0)} - \frac{1}{\theta(\lambda)} = \int_0^\lambda \frac{1}{\theta^2(\lambda')}\frac{d\theta}{d\lambda'}\,\mathrm{d}\lambda' \leq -\frac{1}{2}\lambda\,. \tag{A.9}$$

We summarize these important observations into the following theorem [23].

Theorem A.7 (Focusing theorem)**.** *Let us have a null geodesic congruence $\mathcal{C}$ with the associated tangent vector field k^a. We further assume that k^a is hypersurface orthogonal and that the Einstein equations (with $\Lambda = 0$) together with the null energy condition hold. Then we have (A.8) and if the expansion θ of $\mathcal{C}$ takes a negative value $\theta(0)$ at any point on a geodesic in $\mathcal{C}$, it follows that $\theta \to -\infty$ along the geodesic within the affine length $\lambda \leq \frac{2}{|\theta(0)|}$.*

A.4 Horizons

In this work, various forms of the general notion of horizon make their appearance. Often, an important function of these terms is to generalize the definition of an event horizon, which is as follows [26].

Definition A.8. *Let us have an asymptotically flat spacetime (M, g_{ab}). We define a* ***black hole*** *as a region*

$$\mathcal{B} := M \setminus I^-(\mathscr{I}^+)\,,$$

where $\mathscr{I}^+$ is the future null infinity. The ***event horizon*** *of a black hole is then $\partial\mathcal{B}$.*

This also provides us with a precise definition of a black hole, which is the recurring theme of this work. Notice that this definition is global, as we require an asymptotic region with respect to which it can be formulated. Locating an event horizon locally has proven notoriously tricky.

The second type of horizon that we define is connected to symmetries of our spacetime.

Definition A.9. *A null hypersurface $\mathcal{K}$, with a normal Killing vector field ξ^a that is null on $\mathcal{K}$, is called a* ***Killing horizon****. We further define the* ***surface gravity*** *κ of $\mathcal{K}$ as*

$$\kappa = \sqrt{-\frac{1}{2}\nabla_a \xi_b \nabla^a \xi^b}\bigg|_{\mathcal{K}} . \tag{A.10}$$

Here the name surface gravity is well motivated by the study of black holes. We have seen in Section 1.2 that in the case of a Schwarzschild black hole κ is exactly the Newtonian value of gravitational acceleration. In order to generalize the notion of gravitational acceleration in a relativistically invariant way, one could, for example, consider the norm of four-acceleration of static observers at the horizon (for a Schwarzschild black hole). But from 1.19 we see that $a(r) \xrightarrow{r \to r_s^+} +\infty$. So we consider the acceleration as seen by observers in the asymptotic region. For these observers it is redshifted by the norm of the corresponding Killing vector field, which, by the very definition of the Killing horizon, goes to 0 at the horizon and thus cancels out the divergence. Note that since in the case of a Schwarzschild spacetime static observers move on integral curves of the corresponding Killing vector field, their four-velocity is given only in terms of this vector field and hence the same is true for the four-acceleration. From this we see that in this specific example we have expressed everything only in terms of the given Killing vector field, and therefore everything can be readily generalized to arbitrary Killing horizons (even though in general one cannot physically interpret κ in the same way).

The sense in which the Killing horizon is a generalization of the notion of an event horizon can be made precise. This is attributed to the so-called rigidity theorems by Carter and Hawking. To formulate the essence of these we need to define the most common spacetime symmetries which will also be useful elsewhere.

Definition A.10. *Let us have an asymptotically flat spacetime. We say that it is*

- ***stationary*** *if it admits a Killing vector field t^a that is asymptotically timelike.*
- ***static*** *if it is stationary and if t^a is hypersurface orthogonal.*
- ***axisymmetric*** *if there exists a Killing vector field ϕ^a that generates a one-parameter group of isometries, behaving like a rotation in the asymptotic region.*
- ***t-ϕ orthogonal*** *if it is stationary, axisymmetric and the planes spanned by t^a and ϕ^a are orthogonal to a family of 2-dimensional surfaces*

By a vector field being hypersurface orthogonal, we mean that there exists a foliation of the spacetime such that the vector field is normal to all of its hypersurfaces. By Frobenius's theorem, this is equivalently expressed [86] for a vector field ξ^a by

$$\xi \wedge d\xi = 0 . \tag{A.11}$$

Now to the theorem that summarizes the rigidity theorems (see also [86]).

Theorem A.11 (rigidity)**.** *Let us have an asymptotically flat spacetime with a black hole $\mathcal{B}$. Furthermore, we assume one of the conditions below holds.*

- *The spacetime is static, or it is stationary, axisymmetric and t-ϕ orthogonal. (Carter)*
- *The spacetime is stationary with electrovacuum and Einstein-Maxwell equations hold. (Hawking)*

Then it follows that $\partial\mathcal{B}$ is a Killing horizon for some Killing vector field ξ^a. In the case of static spacetime $\xi^a = t^a$ and for stationary spacetime $\xi^a = t^a + \omega\phi^a$ where $\omega \in \mathbb{R}$.

The trapped surface and the apparent horizon are also important notions used to approximately locate the event horizon locally. We can define the trapped surface as a surface inside which is even light trapped due to spacetime curvature. The apparent horizon is then in some sense the outermost of the trapped surfaces.

Definition A.12. *Let us have spacetime (M, g_{ab}) with a compact, orientable, spacelike, two-dimensional surface $S \subset M$. We say that S is a **trapped surface** if for both congruences of null geodesic curves orthogonal to Σ, it holds that $\forall p \in \Sigma : \theta(p) < 0$.*

*If there exists a Cauchy surface $\Sigma \subset M$. We define a **trapped region** in Σ as the set $\mathcal{T} = \{p \in \Sigma \,|\, \exists$ trapped surface $S : p \in S\}$. The **apparent horizon** is then $\partial\mathcal{T}$.*

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